Optimal Portfolio Selection: the Role of Illiquidity and Investment Horizon

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Abstract

Modern Portfolio Theory is a single-period model developed for the efficient securities market, in which asset prices are implicitly assumed to follow a random walk. It is widely agreed that real estate does not fit into the efficient market paradigm; however, mixed-asset portfolio analysis continues to rely on Modern Portfolio Theory. This paper proposes an alternative portfolio theory that extends the classical Modern Portfolio Theory to accommodate multi-period utility maximization as well as the unique characteristics of real estate such as liquidity risk, horizon-dependence of real estate returns and high transaction cost. Using real world data, we find that the optimal allocation to real estate in the mixed-asset portfolio is much lower than previously suggested by the literature, and is quite in line with the reality of institutional portfolios.

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1. Introduction

The appropriate role of private real estate in mixed-asset portfolios is a long standing puzzle in the real estate literature. Despite the general consensus among academics and practitioners that private real estate investment can bring additional diversification benefits to the traditional security-only portfolios, wide disagreement remains with regard to the optimal allocation to real estate. For example, while numerous academic studies since the 1980s have repeatedly suggested that real estate should constitute 15% to 40% or more of a diversified portfolio,\(^1\) leading institutional investors typically have only about 3% to 5% and almost never more than 10% of their total assets in real estate.\(^2\) Significant effort has been devoted to resolving the discrepancy, but to date there appears to be no agreeable consensus among academics. The current paper attempts to resolve this old issue with a new approach.

The elegance of Modern Portfolio Theory is based on some critical assumptions. For example, the theory premises on a centralized and well-functioning asset market in which trading

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\(^1\) See, for example, Hartzell, Hekman and Miles (1986), Fogler (1984), Firstenberg, Ross and Zisler (1988), and Hudson-Wilson, Fabozzi and Gordon (2005), among others.

\(^2\) See, for example, a survey of 173 institutional investors reported in Goetzmann and Dhar (2005).
is continuous and frictionless, and the complex characteristics of the assets can be conveniently captured with two parameters – the means and variances of their expected returns. Real estate, however, are infrequently traded in decentralized private markets that are highly inefficient, and their returns are neither normal nor stable over time. (Young and Graff (1995), Graff, Harrington and Young (1997), Young, Lee and Devaney (2006), and Young (2008)) Secondly, non-variance factors – illiquidity, immobility, non-divisibility, etc. are unique to real estate and the traditional mean and variance are inadequate in capturing the impact of these risk factors on real estate price. (Lusht (1988)) Lastly, while financial assets can be inexpensively traded with ease, the high transaction cost of real estate prevents investors from trading the asset frequently, which implies that real estate has to be held for a much longer period to be a viable investment. Most researchers acknowledge that these features of real estate do not conform with the finance paradigm, but in the absence of an alternative portfolio theory, the issues are often dismissed or downplayed as minor deviations. As a result, mixed-asset portfolio analysis continues to rely on Modern Portfolio Theory.

Previous effort in resolving the real estate allocation puzzle has largely been focused on attempting to “fine-tune” the application of Modern Portfolio Theory (MPT), by either imposing additional constraints that limit the maximum weights on real estate, and/or, in case the NCREIF data are used, inflating the real estate risk by some de-smoothing procedure. To a large extent, though, these ad hoc solutions seem to have confounded the puzzle further rather than solving it.

The current study takes a different approach – instead of tweaking the way the theory is applied, we modify the theory itself. Our objective is to develop a formal framework that extends the classical MPT to accommodate the unique characteristics of real estate. Specifically,
our analysis explicitly incorporates three most distinctive characteristics of real estate: (1) real estate returns are horizon-dependent; (2) real estate bears liquidity risk; and (3) real estate involves high transaction costs. Although these characteristics have been extensively studied in the literature in various contexts, the accumulated knowledge has yet to be synthesized to yield a formal model for mixed-asset portfolio analysis. Building upon a series of recent research in this area, this paper attempts to fill the void. We first develop an alternative model for the mixed-asset portfolio and then apply the model to real world data. Without resorting to any de-smoothing procedure or imposing any ad hoc constraints, we find that the optimal allocations to real estate in the mixed-asset portfolio are substantially lower than previously suggested in the literature and are more consistent with the observed practice of institutional investors.

2. Investment horizon, illiquidity, and transaction cost of real estate

2.1 Horizon-dependence of real estate performance

Modern Portfolio Theory is essentially a single-period model, which assumes that assets in the portfolio are to be held for “one period,” and the optimal portfolio is the one that maximizes the investor’s objective over such a single-period horizon. The validity of the theory to the multi-period investment reality depends on a critical assumption – all asset returns are independent and identically-distributed (i.i.d.) over time. Early studies by Merton (1969), Samuelson (1969), and Fama (1970) have shown that, under the i.i.d. condition, investor’s utility maximization over multiple periods are indistinguishable from that over a single period, that is, investment horizon is irrelevant. This is a powerful finding because it effectively implies that portfolio optimization only needs to be based on the assets’ single-period performance, regardless of investor’s expected investment horizon. The importance of the i.i.d. condition,
therefore, should not be underestimated. It is the critical link that bridges the gap between the single-period theory and the multi-period investment reality.

Such a link does not exist for real estate. In fact, numerous studies have repeatedly documented that real estate returns exhibit strong auto-correlation and serial persistence, that is, they are not i.i.d. from period to period. But for some reason, this knowledge is often perceived as being perhaps too basic and encompassing to be practically useful, and it has typically not prevented researchers from treating real estate just like other financial assets in portfolio analysis. Some would argue that the assumption can be adopted for convenience because, after all, the stock returns are probably not perfectly i.i.d. either. But given the critical importance of i.i.d. to the traditional MPT application, two questions naturally arise: First, can real estate returns be considered “reasonably close” to the i.i.d. condition? Second, if not, what is the alternative distribution for real estate returns? Recent literature has addressed these issues.

Cheng, Lin, Liu and Zhang (2011) attempt to shed light on the first question by conducting a direct test of the i.i.d. hypothesis on a wide variety of asset classes including common stocks, private real estate, and REITs. Using the widely regarded BDS test by Brock, Dechert and Scheinkman (1987), they find that, while financial assets are reasonably close to the i.i.d. assumption, the null hypothesis of the i.i.d. condition is strongly rejected by real estate assets.

The non-i.i.d. nature of real estate returns implies that real estate performance is horizon-dependent. In other words, the performance of multi-period real estate investment cannot be properly measured by single-period performance metrics. To large extent, this is also true for

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3 The literature on the subject is too large to be reviewed here fully. A few examples include Case and Shiller (1989), Young and Graff (1995), Englund, Gordon and Quigley (1999) and Gao, Lin and Na (2009), among others.
some non-real estate portfolios as well. For example, in a study of hedge fund performances, Clifford, Krail and Liew (2001) find that simple estimate of volatility using monthly returns understates the actual fund volatility, and simple monthly beta and correlation greatly underestimate the fund’s equity market exposure. They attribute the cause of such biased monthly performance measures to the fact that hedge funds typically hold illiquid exchange-traded securities or over-the-counter securities that do not trade on a monthly basis, but are held for much longer period of time. They argue that the appropriate performance measure should coincide with the typical holding period of the assets.

So how does investment horizon affect the return and variance of multi-period real estate investment? Lin and Liu (2008) formally model the real estate transaction process and propose an alternative assumption that the variance of real estate return increases with the square of the holding time (as opposed to linearly increasing with holding time under the i.i.d. assumption). They further show that residential real estate data is more consistent with this alternative assumption than with the i.i.d. assumption. In a subsequent study, Cheng, Lin and Liu (2010a) extend the examination to the commercial real estate market and find that the risk structure in the commercial market is even closer to the alternative assumption than the residential market. In a related study, Cheng, Lin, and Liu (2010b) further argue that, while real estate is closer to the alternative assumption of Lin and Liu (2008) than to the i.i.d. assumption, the reality lies in between and can be in fact estimated empirically. The empirical approach proposed by Cheng, Lin, and Liu (2010b) can be briefly described as follows:

Cheng, Lin, and Liu (2010b) use the quarterly NCREIF Property Indices (NPI) for the four major commercial property sectors – office, apartment, industrial, and retail, as well as the “all property” index, spanning from 1978Q1 to 2007Q2. Since the NPI is reported in quarterly
returns, they convert the entire series into an index with 1977Q4 being 100. They then repeatedly using the index to compute the holding-period returns for a range of possible holding period. For example, for a holding period of four quarters (or one year), starting at the beginning of the index, investors “buy” the index in the first quarter, and then “sell” at four quarters later, this produces one return observation for a 1-year holding period. Investors then move down the index to “buy” at every quarter and “sell” four quarters later until the index is exhausted. Such “rolling window” calculation is then repeated for every holding period under consideration, ranging from 1 quarter to 36 quarters (or 9 years). Their choice of 36 quarters (or 9 years) is mainly due to the limited length of the NPI series. As more data become available in the future, the range of holding period can readily be extended to more than 9 years. However, they argue that a 9-year holding period is not unrealistic for commercial properties, since several studies have shown that many institutional investors hold commercial properties for about five to eight years.

Once we have computed all the returns for each holding period (displayed in the columns of Appendix I), we can compute the standard deviations of these holding-period returns. Finally, we standardize the standard deviations by scaling them in such a way so that the standard deviation for holding one-quarter is 1.00. The “scaled” standard deviations are then plotted in the Figure 1. For comparison, we then replicate the entire calculations of Appendix I for all four sub-indices and the S&P500. Furthermore, given the “scaled” standard deviation for a single-quarter holding period being 1, we also plot the multi-period risk increasing along the path of i.i.d. condition \( (\sigma_\tau / \sigma_1 = \sqrt{\tau}, \text{ where } \sigma_1 \text{ is the risk of single-period returns and } \tau \text{ is the holding period} \).

\(^4\) The detailed calculation is illustrated in Appendix I. To save space, not all 36 quarters of computation are displayed.
period), as well as the alternative risk structure proposed by Lin and Liu (2008) (e.g. \( \sigma_r / \sigma_1 = \tau \)). Cheng, Lin, and Liu (2010b) call the lines in Figure 1 the *risk curves*, by analogy to the well-known *yield curves*.

Figure 1. The Risk Curves of NCREIF Indices (1978Q1 – 2007Q2)

Figure 1 shows that, while the S&P 500 can be seen as being reasonably close to the i.i.d. assumption, the risk curves of NCREIF indices are much closer to the alternative assumption, suggesting that real estate is not i.i.d. On the other hand, while the alternative assumption may
be reasonable approximations for some NCREIF indices, it is less so for others. A more general approach, as proposed by Cheng, Lin, and Liu (2010b), is to empirically estimate the risk curves.

To do this, notice that all lines in Figure 1 passes through the point (1,1). Suppose that the slope of the line is $\beta$, we thus have

$$\frac{\sigma_\tau}{\sigma_1} - 1 = \beta(\tau - 1) \quad (1)$$

Or

$$\sigma_\tau = [(1 - \beta) + \beta \tau] \sigma_1 \quad (2)$$

where $\tau$ is the holding period, $\sigma_\tau$ is the risk of the holding-period returns and $\sigma_1$ is the risk of single-period returns. When $\beta = 1$, Equation (2) becomes the alternative assumption, $\sigma_\tau = \tau \sigma_1$.

For the NCREIF indices, Cheng, Lin, and Liu (2010b) estimate $\beta$ in the range of 0.82 to 0.96. Combining this with the fact that real estate typically have long holding period of multiple years, $(1 - \beta)$ in Equation (2) should be relatively small as opposed to $\beta \tau$. For mathematical simplicity, we use the following approximation:

$$\sigma_\tau \approx \beta \tau \sigma_1 \quad (3)$$

Note that for $\beta = 1$, Equation (3) becomes the alternative assumption.

With the empirical estimate of Equation (3), the return distribution of real estate assets can be expressed as follows, 6

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5 In fact, Cheng, Lin, and Liu (2009, 2010b) have examined a wider range of real estate indices and sub-indices, both commercial and residential, and found that all real estate risk curves lie in between the two assumptions and the vast majority of them are much closer to the alternative assumption than to the i.i.d. assumption.

6 Cheng, Lin, and Liu (2010b) find that the average periodic returns for the NCREIF data are about the same across different holding periods. As a result, the total holding-period return $u_\tau = \tau u$ in Equation (4) is reasonably accurate.
\[ u_r = \mu \]
\[ \sigma_r = \beta \sigma \]  

(4)

It is worth noting that the findings in Figure 1 are in contrast with those of Rehring (2011), ManKinnon and Zaman (2009) and Pagliari (2011), which suggest the variance of real estate returns declines or “decays” in the long run. The cause of the difference may be in their approaches to the question. Whereas the above three studies all rely on return-generating models based on certain assumptions on auto-correlation, mean-reversion, and so on, Cheng, Lin, Liu (2010b) observe the historical data as-is, and directly compute multi-period return series from the data without using any return-generating models. Their finding is consistent with an earlier study by Collett, Lizieri and Ward (2003) who directly observe from the U.K. commercial data reaches the similar conclusion that “even allowing for appraisal smoothing, private real estate looks less attractive once more realistic and longer time horizons are considered”. (p. 207)

2.2 Illiquidity and \textit{ex ante} risk of real estate

Proper estimate of the real estate risk, however, requires more than incorporating the impact of holding period, because real estate is subject to another important risk – liquidity risk. Real estate cannot be easily bought and sold at any time an investor desires. The potential loss of welfare due to such an inability to trade out of a position when needed is a significant source of risk and must be properly accounted for by rational investors. Real estate investors have been dissatisfied with the way the performance is currently measured. A fairly recent survey by Goetzmann and Dhar (2005) reports that majority of the surveyed institutional portfolio managers consider illiquidity as their number one risk factor for real estate investment, and the
lack of appropriate performance metric that quantifies such risk in formal analysis is a major challenge to their portfolio decision-makings. This concern is justified for two reasons:

First, liquidity risk is real and substantial in *ex ante*. Portfolio decisions are forward-looking in nature. Proper asset performances, therefore, should be measured with *ex ante* rather than *ex post* metrics. This is particularly important for real estate because, at the time of the acquisition decision, the future sale at the end of the holding period for real estate is very different from that for financial assets. While a stock investor only faces the uncertain sales price, real estate investor faces an additional uncertainty—the uncertain and lengthy time-on-market (TOM). According to the data from the National Association of Realtors, the average TOM in the U.S. residential market during the period from 1989 to 2006 is about 6 months, and it is substantially longer in the commercial property market. While liquidity risk may be mitigated by longer holding period, most properties are not held for very long in practice. As suggested by Gau and Wang (1994) and Collett, Lizieri and Ward (2003), the typical holding period is about 5 – 9 years for commercial properties. Within this range, Cheng, Lin, and Liu (2010a) find that the liquidity risk alone still contribute an additional 6% to 29% to the total *ex ante* risk of quarterly NCREIF index. The notion that liquidity risk is negligible in the long run is not entirely correct.

Second, liquidity risk is a systematic risk. Generally speaking, the expected TOM is a function of market conditions, and is not under the full control of the seller. In hot markets all properties are sold rather quickly, while in cold markets the average TOM will be substantially longer. Individual sellers may be able to influence their individual selling time with listing strategies subject to their financial constraints, but they cannot control the average TOM under a given market condition. A quick sale typically results in significant price discount from the
property’s fair value. While the (deep) discount may reflect the degree of financial distress of the seller, it does not properly reflect the trading strategy of normal sellers who would take the time necessary under given market conditions and search until a desirable offer arrives. In other words, liquidity risk is priced by the market. Conventional risk measures which ignore liquidity risk fail to account for this component of systematic risk.

Being part of systematic risk, illiquidity cannot be diversified away in a portfolio. Although Bond, et al. (2007) suggest that the risk related to uncertain time-on-market can be diversified away by constructing a portfolio of ten or more properties, a subsequent study by Lin, Liu and Vandell (2009) shows that this is true only under the i.i.d. assumption. Otherwise they show that the risk due to uncertain time-on-market does not in general approach zero in the limit, in fact could increase or decrease depending upon the illiquidity characteristics of the individual assets and the correlation among individual property returns and marketing periods. They conclude that “even large institutional real estate portfolio managers must consider the illiquidity present in their portfolios and cannot assume that its effect will be diversified away”. (p. 191)

2.3 Transaction cost and holding period

Real estate involves high transaction costs. Using U.K. data, Collett, Lizieri and Ward (2003) report “the round-trip lump-sum costs” were approximately 7-8% of the asset value. It should be noted that the effect of high transaction cost is more than just lowering the net sales price or returns, it also prevents frequent trading and affects investment horizon, which in turn affects the risk of real estate investment. Modern Portfolio Theory does not consider this effect of transaction cost. But a number of studies have documented that, in the presence of transaction cost, single-period utility maximization is not the same as multi-period utility maximization. Mayshar (1979) incorporates fixed transaction cost into a single-period mean-variance model
and finds that even a small transaction cost results in investors holding fewer assets (as opposed to the market portfolio) for longer periods, which implies that the Sharpe-Lintner CAPM is no longer valid. Constantinides (1986) finds that proportional transaction cost substantially reduces demand for assets and increases their holding periods. He also finds that a single-period model cannot simply be extended to multiple periods, because the appropriate holding periods are asset-specific. Amihud and Mendelson (1986) develop a theoretical model that predicts that higher bid–ask spreads (proxies for transaction costs) are correlated with longer holding periods. Their finding is supported by the empirical evidence in Atkins and Dyl (1997), who find significant correlations between holding period and the bid-ask spreads. Collectively, these studies imply that the classical single-period MPT is not appropriate for multi-period real estate investment because of high transaction cost.

In summary, the complex nature of real estate, as well as the complexity of its performance measurement, suggests that mixed-asset portfolio theory will be more complex than the classical MPT. The horizon-dependent performance, liquidity risk, and high transaction cost of real estate implies that optimal diversification of mixed-asset portfolio should be based on multi-period utility maximization. In the next sections, we present an alternative model that extends the classical MPT into a multi-period model, and demonstrate its application using real world data. Our results indicate that real estate allocation turns out to be much lower than that the MPT suggests, thus reconciling the gap between academics and practitioners with regard to the decades-old real estate allocation puzzle.
3. The Importance of i.i.d. Condition to Modern Portfolio Theory

A basic premise of Modern Portfolio Theory is that investors are rational and risk averse. They like returns but dislike risk. The optimal portfolio decision, therefore, is to seek the efficient portfolio that (a) has the highest expected return for a given level of risk, or (b) has the minimum risk for a desired level of expected return. Since these are equivalent optimization problems, one way to solve the problem is to maximize the expected return for a given level of risk. This is commonly known as the mean-variance analysis.\(^7\) A general representation of the mean-variance analysis can be described as follows.

Consider a simple world where there are \( N \) classes of assets available for investment. Suppose that an investor’s investment horizon is \( T \) periods. At any given level of risk \( \Sigma^2 \), the investor’s objective is to choose an optimal weight \( w_i \) on asset \( i \) (\( i = 1, 2, 3, \ldots, N \)) to satisfy the following:

\[
\max_{(w_1, w_2, \ldots, w_N, \sum w_i = 1)} \left\{ E \left[ \sum_{i=1}^{N} w_i \tilde{R}_{i,T} \right] \right\} \\
\text{St. } \text{Var} \left( \sum_{i=1}^{N} w_i \tilde{R}_{i,T} \right) = T\Sigma^2
\]

where \( \tilde{R}_{i,T} \) is the total return on individual asset \( i \) over \( T \) periods and \( E \left[ \sum_{i=1}^{N} w_i \tilde{R}_{i,T} \right] \) is the expected holding-period (total) return of the portfolio. Mathematically, (5) is equivalent to solving the following Lagrangian function,

\(^7\) Modern Portfolio Theory was developed by Markowitz (1952), Sharpe (1963), and Brennan (1975), and among others.
For a given level of risk, there exists a unique \( \lambda \) in Lagrangian function (6), which \( \lambda \) essentially captures the degree of the investor’s risk aversion. Mathematically, Lagrangian function (6) is equivalent to maximizing the portfolio’s risk-adjusted return for a given risk aversion parameter \( \lambda \), i.e.,

\[
\max_{(w_1, w_2, \ldots, w_N \sum w_i = 1)} \left\{ E\left[ \sum_{i=1}^{N} w_i \tilde{R}_{i,T} \right] - \lambda \left( Var\left( \sum_{i=1}^{N} w_i \tilde{R}_{i,T} \right) - T \Sigma^2 \right) \right\}.
\]

Generally speaking, the optimal weights \( (w_1^*, w_2^*, \ldots, w_N^*) \) in (6) are a function of holding period \( T \). That is, they are holding-period dependent. Suppose that \( \tilde{R}_{i,T} = \sum_{t=1}^{T} \tilde{r}_{i,t} \), where \( \tilde{r}_{i,t} \) is the return on asset \( i \) in a single period \( t \), and assume that the i.i.d. condition is only partially violated by \( \tilde{r}_{i,t} \). In particular, \( \tilde{r}_{i,t} \) is independent but NOT identically distributed over time. Then the mean and variance of \( \tilde{r}_{i,t} \) are time-varying, i.e., \( E(\tilde{r}_{i,t}) = u_i(t) \) and \( Var(\tilde{r}_{i,t}) = \sigma_i^2(t) \) \( (t = 1, 2, \ldots, T) \).

Furthermore, we denote the correlation between \( \tilde{r}_{i,t} \) and \( \tilde{r}_{j,t} \) as \( \rho_{ij} \), we then have,

\[
Var\left( \sum_{i=1}^{N} w_i \tilde{R}_{i,T} \right) = Var\left( \sum_{i=1}^{N} w_i \sum_{t=1}^{T} \tilde{r}_{i,t} \right)
= \sum_{i=1}^{N} w_i^2 \left( \sum_{t=1}^{T} \sigma_i^2(t) \right) + \sum_{t=1}^{T} \left[ \sum_{i=1}^{N} w_i \rho_{ij} \sigma_i \sigma_j(t) \right]
\]

\[
E\left[ \sum_{i=1}^{N} w_i \tilde{R}_{i,T} \right] = E\left[ \sum_{i=1}^{N} w_i \sum_{t=1}^{T} \tilde{r}_{i,t} \right]
= \left[ \sum_{i=1}^{N} w_i \left( \sum_{t=1}^{T} u_i(t) \right) \right]
\]
It can be seen that, as long as the expected return $u_i(t)$ and variance $\sigma^2_i(t)$ vary from period to period, (7) and (8) cannot be further simplified and the optimal solution to the Lagrangian function (6) will be horizon-dependent. However, if $\tilde{r}_{i,j}$ is not only independent, but also identically-distributed, that is, $\tilde{r}_{i,j}$ is i.i.d. over time, we have $u_i(t) = u_i$ and $\sigma^2_i(t) = \sigma^2_i$ for all $t = 1, 2, \ldots, T$. Thus (7) and (8) can be further simplified as

$$Var \left( \sum_{j=1}^{N} w_j \tilde{R}_{i,T} \right) = T \left[ \sum_{i=1}^{N} w_i^2 \sigma^2_i + \sum_{i,j=1, i \neq j}^{N} w_i w_j \rho_{ij} \sigma_i \sigma_j \right]$$

(7’)

$$E \left[ \sum_{i=1}^{N} w_i \tilde{R}_{i,T} \right] = T \sum_{i=1}^{N} w_i u_i$$

(8’)

Notice that now the holding period $T$ is moved outside of the summation operations. Given that

$$Var \left( \sum_{i=1}^{N} w_i \tilde{r}_i \right) = \left[ \sum_{i=1}^{N} w_i^2 \sigma^2_i + \sum_{i,j=1, i \neq j}^{N} w_i w_j \rho_{ij} \sigma_i \sigma_j \right]$$

and

$$E \left[ \sum_{i=1}^{N} w_i \tilde{r}_i \right] = \sum_{i=1}^{N} w_i u_i$$

we can rewrite (7’) and (8’) and then substitute them into (6) to obtain

$$L(w_1, w_2, \ldots, w_N, \sum_{i=1}^{N} w_i = 1, \lambda) = T \left\{ E \left[ \sum_{i=1}^{N} w_i \tilde{r}_i \right] - \lambda (Var(\sum_{i=1}^{N} w_i \tilde{r}_i) - \Sigma^2) \right\}$$

(9)

For any given holding period $T$, (9) is equivalent to maximizing the expected return given a level of risk for holding only one period. In other words, under the i.i.d. condition, the multi-period mean-variance optimization in (9) is equivalent to the single-period mean-variance optimization.

$$\max_{(w, w_2, \ldots, w_N, \sum_{i=1}^{N} w_i = 1)} \left\{ E \left[ \sum_{i=1}^{N} w_i \tilde{r}_i \right] \right\}$$

(10)
St. $\text{Var} \left( \sum_{i=1}^{N} w_i \tilde{r}_i \right) = \Sigma^2$

Therefore, under the i.i.d. condition, the holding period $T$ is irrelevant to the choice of the optimal portfolio. In fact, (10) is the familiar textbook presentation of the Modern Portfolio Theory.

To analyze a simple mixed-asset portfolio, suppose that there are only three assets available to investors: An illiquid real estate asset, a liquid financial asset and a risk-free asset. By directly applying Modern Portfolio Theory, solving (10) is equivalent to solving the following Lagrangian function:

$$\max_{w_{RE}, w_L} \left\{ w_{RE} u_{RE} + w_L u_L + (1 - w_{RE} - w_L) r_f - \lambda \left[ w_{RE}^2 \sigma_{RE}^2 + 2 w_{RE} w_L \sigma_{RE} \sigma_L \rho + w_L^2 \sigma_L^2 \right] \right\}$$

(11)

where

- $u_{RE}$ and $\sigma_{RE}$ are the single-period return and risk of real estate asset;
- $u_L$ and $\sigma_L$ are the single-period return and risk of the financial asset;
- $\rho$ is the correlation between the real estate and financial asset;
- $r_f$ is the single-period return for the risk-free asset;
- $w_{RE}$ and $w_L$ are the weights on the real estate and financial assets, respectively;
- $\lambda$ is investor’s risk-averse parameter.

Therefore, we can calculate the optimal weight on the real estate as follows,
Based on the Modern Portfolio Theory, a number of studies such as Hartzell, Hekman and Miles (1986), Fogler (1984), Firstenberg, Ross and Zisler (1988), and Hudson-Wilson, Fabozzi and Gordon (2005), among others, have suggested that any diversified portfolio should contain 15% to 40% in real estate. Obviously, these results are based on the implicit (and inappropriate) assumption that real estate returns are i.i.d over time. In the next section, we present an alternative model that extends the MPT to incorporate three distinct features of real estate: (1) the horizon-dependent of its returns; (2) liquidity risk; and (3) high transaction cost.

4. An Alternative Model for the Mixed-asset Portfolio

Our central task is to extend the Modern Portfolio Theory to accommodate multi-period utility maximization as well as the unique features of real estate. To begin, we first examine the chain of events occurring in the sale of the three-asset portfolio which contains a liquid financial asset, an illiquid real estate, and a risk-free asset. As illustrated in Figure 2, suppose that the investor acquires a portfolio at time 0 and wants to sell it after holding $T$ periods. The financial asset (and the risk-free asset) can be sold immediately at time $T$, but the real estate will have to wait to be sold at $T + \tilde{t}$, where time-on-market ($\tilde{t}$) and the eventual sales price are both uncertain and not fully within the control of the investor. Note that the investor’s actual holding

$$w^\ast_{RE} = \frac{u_{RE} - r_f}{\sigma_{RE}} - \rho \left( \frac{u_L - r_f}{\sigma_L} \right) \frac{2\lambda(1 - \rho^2)\sigma_{RE}}{2\lambda(1 - \rho^2)\sigma_{RE}}$$ (12)

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8 For mathematical simplicity, we assume that there is no partial sale.

9 Immediate sale of real estate (as in the case of imminent foreclosure) would characterize the behavior of an extremely distressed seller. While the likely (deep) price discount reflects the degree of distress of the seller, it does
period for the real estate is $T + \tilde{t}$ and $\tilde{t}$ is random. Without specifying a particular distribution of time-on-market $\tilde{t}$, we denote its mean and variance as $t_{TOM}$ and $\sigma^2_{TOM}$, respectively. Both $t_{TOM}$ and $\sigma^2_{TOM}$ essentially capture the degree of real estate illiquidity. In addition, suppose that real estate incurs lump-sum cost ($C$).

Figure 2. The Sequence of Events in the Sale of the Portfolio

(1) Place the portfolio in the market

0 ------------------- $T$ ----------------------------- $\tilde{t}$ (random) -------

Time on market

(2) Sale of the financial asset

(3) Sale of the real estate asset with lump-sum cost: $C$

Based on Figure 2, we can express the optimal asset allocation to the mixed-asset portfolio by solving the following optimization problem,

$$
\max_{w_L, w_L} \left\{ E^{ex-ante}\left[ \tilde{R}^{\text{Portfolio}}_{T+\tilde{t}} \right] - \lambda Var^{ex-ante}\left( \tilde{R}^{\text{Portfolio}}_{T+\tilde{t}} \right) - w_{RE} C \right\} 
$$

(13)

where the portfolio return $\tilde{R}^{\text{Portfolio}}_{T+\tilde{t}}$ is a weighted average of the three assets’ returns, i.e.,

$$
\tilde{R}^{\text{Portfolio}}_{T+\tilde{t}} = w_{RE} \tilde{R}^{RE}_{T+\tilde{t}} + w_{L} \tilde{R}^{L}_{T+\tilde{t}} + (1 - w_{RE} - w_{L}) \tilde{T}r_f.
$$

(14)

To compute the $ex$ $ante$ mean and variance, $E^{ex-ante}\left[ \tilde{R}^{\text{Portfolio}}_{T+\tilde{t}} \right]$ and $Var^{ex-ante}\left( \tilde{R}^{\text{Portfolio}}_{T+\tilde{t}} \right)$, we need to know the distributions of both real estate and financial returns. For the financial asset, the
returns over time can be reasonably assumed to follow the i.i.d. assumption. For the real estate asset, the returns can be illustrated in Equation (4). Therefore, we have,

For the financial asset, \( E[\tilde{R}_T^L] = Tu_L \) and \( \text{Var}(\tilde{R}_T^L) = \sqrt{T} \sigma_L^2 \)

For the real estate asset, \( E[\tilde{R}_{T+i}^RE | T + \tilde{t}] = (T + \tilde{t})u_{RE} \) and \( \text{Var}(\tilde{R}_{T+i}^RE | T + \tilde{t}) \approx \beta(T + \tilde{t}) \sigma_{RE} \)

Given (15), we can first calculate the expected \textit{ex post} return and then use the Law of Iterated Expectations:

\[
E^{\text{ex-ante}}[\tilde{R}_{T+i}^\text{Portfolio}] = E[E[\tilde{R}_{T+i}^\text{Portfolio} | T + \tilde{t}]]
\]

Simplifying Equation (16) yields:

\[
E^{\text{ex-ante}}[\tilde{R}_{T+i}^\text{Portfolio}] = w_{RE} (T + t_{TOM})u_{RE} + w_LTu_L + (1 - w_{RE} - w_L)Tr
\]

where \( t_{TOM} = E[\tilde{t}] \).

The \textit{ex-ante} variance of portfolio returns \( \text{Var}^{\text{ex-ante}}(\tilde{R}_{T+i}^\text{Portfolio}) \) can be computed by applying the conditional variance formula for two stochastic variables as follows,

\[
\text{Var}^{\text{ex-ante}}(\tilde{R}_{T+i}^\text{Portfolio}) = \text{Var}[E[\tilde{R}_{T+i}^\text{Portfolio} | T + \tilde{t}]] + E[\text{Var}(\tilde{R}_{T+i}^\text{Portfolio} | T + \tilde{t})]
\]

The first term on the right-hand side of Equation (18) is

\[
\text{Var}[E[\tilde{R}_{T+i}^\text{Portfolio} | T + \tilde{t}]] = \text{Var}[w_{RE} (T + \tilde{t})u_{RE} + w_LTu_L + (1 - w_{RE} - w_L)Tr]
\]

To further simply Equation (19), we can obtain,

\[
\text{Var}[E[\tilde{R}_{T+i}^\text{Portfolio} | T + \tilde{t}]] = w_{RE}^2 u_{RE}^2 \sigma_{TOM}^2
\]

where: \( \sigma_{TOM}^2 = \text{Var}(\tilde{t}) \). The second term on the right-hand side of Equation (18) can be expressed as
\[ E[\text{Var}(\tilde{R}_{\text{Portfolio}}|T + \tilde{t})] = E[(w_{RE}\beta(T + \tilde{t})\sigma_{RE})^2 + w_L^2T\sigma_L^2 + 2\rho w_{RE}w_L\beta T^2\sigma_{RE}\sigma_L] \]  \hspace{1cm} \text{(21)}

Given that \( E[(w_{RE}\beta(T + \tilde{t})\sigma_{RE})^2] = w_{RE}^2\beta^2\sigma_{RE}^2E[T + \tilde{t}]^2 \) and \( E[T + \tilde{t}]^2 = \sigma_{TOM}^2 + (T + t_{TOM})^2 \), we can rewrite Equation (21) as follows,

\[ E[\text{Var}(\tilde{R}_{\text{Portfolio}}|T + \tilde{t})] = w_{RE}^2\beta^2\sigma_{RE}^2[\sigma_{TOM}^2 + (T + t_{TOM})^2] + w_L^2T\sigma_L^2 + 2\rho w_{RE}w_L\beta T^2\sigma_{RE}\sigma_L \]  \hspace{1cm} \text{(22)}

Substituting Equations (20) and (22) into Equation (18), we have

\[ \text{Var}^{\text{ex-ante}}(\tilde{R}_{\text{Portfolio}}|T + \tilde{t}) = w_{RE}^2[(u_{RE}^2 + \beta^2\sigma_{RE}^2)\sigma_{TOM}^2 + \beta^2\sigma_{RE}^2(T + t_{TOM})^2] + w_L^2T\sigma_L^2 + 2\rho w_{RE}w_L\beta T^2\sigma_{RE}\sigma_L \]  \hspace{1cm} \text{(23)}

From Equations (17) and (23), we can rewrite the optimal asset allocation to the mixed-asset portfolio in Equation (13) as follows,

\[ \max_{w_{RE}, w_L} \left\{ \begin{array}{c} w_{RE}(T + t_{TOM})u_{RE} + w_L T u_L + (1 - w_{RE} - w_L) T r_f - w_{RE} C - \lambda \left[w_{RE}^2[(u_{RE}^2 + \beta^2\sigma_{RE}^2)\sigma_{TOM}^2 + \beta^2\sigma_{RE}^2(T + t_{TOM})^2] + w_L^2 T \sigma_L^2 + 2\rho w_{RE}w_L\beta T^2\sigma_{RE}\sigma_L \right] \end{array} \right\} \text{ (24)} \]

The important difference between the mixed-asset optimization problem and Modern Portfolio Theory is that the optimal allocations in (24) \( (w_L^* \text{ and } w_{RE}^*) \) are conditional upon the expected holding period \( T \), whereas the classical MPT is a single-period model. In addition, the model here treats the real estate differently from the financial asset in three aspects: First, it distinguishes the return behavior of real estate from that of financial assets by recognizing the fact that real estate returns over time are not i.i.d.; Second, the real estate performance requires a different measure from that of financial assets: the \textit{ex ante} measure, which is a forward-looking measure unconditional upon a successful sale at a specific point in time; Third, the model explicitly incorporates the high transaction cost of real estate.
Now suppose that real estate can be traded instantly with trivial time-on-market risk, i.e.,
\[ t_{TOM} \approx 0 \] and \[ \sigma_{TOM}^2 \approx 0 \], and further suppose that the transaction cost is negligible when selling
real estate \((C \approx 0)\). The optimization problem (24) can then be rewritten as

\[
\max_{w_{RE},w_L} \left\{ T \left( w_{RE}u_{RE} + w_Lu_L + (1-w_{RE}-w_L)r_f - \lambda \left[ w_{RE}^2 \beta^2 \sigma_{RE}^2 T + w_L^2 \sigma_L^2 + 2\rho w_{RE}w_L \beta T^2 \sigma_{RE} \sigma_L \right] \right) \right\} 
\]

(25)

This is still different from the MPT as in Equation (11) because of the holding period \(T\) in
Equation (25). In other words, as long as we recognize that real estate returns are not i.i.d. over
time, the single period MPT cannot be applicable to the mixed-asset portfolio. Our alternative
model developed in (24) overcomes this limitation and accommodates other features of real
estate as well (i.e. illiquidity risk and transaction costs).

By solving the optimization problem in (24), we can obtain the optimal weigh to real
estate \(w_{RE}^*\) for any investor with regard to his holding period \(T\). While a closed-form formula of
\(w_{RE}^*\) is not easy, some important properties of \(w_{RE}^*\) can be examined. Unlike the MPT, which
suggests that the weight of an asset in a portfolio is a function of \(u_{RE}, \sigma_{RE}, u_L, \sigma_L, \rho, and r_f\), the
optimal weight of real estate \((w_{RE}^*)\) is further affected by the asset’s illiquidity \((t_{TOM}, \sigma_{TOM}^2)\),
degree of non-i.i.d. nature of its returns \((\beta)\), and the transaction cost \((C)\). Specifically, \(w_{RE}^*\) is
negatively related to \(\sigma_{TOM}^2, \beta\) and \(C\), i.e., (a). \(\frac{\partial w_{RE}^*}{\partial \sigma_{TOM}} < 0\); (b). \(\frac{\partial w_{RE}^*}{\partial \beta} < 0\); (c). \(\frac{\partial w_{RE}^*}{\partial C} < 0\).
5. A Numerical Example

In this section, we use a numerical example to examine how real estate illiquidity \( (t_{\text{TOM}}, \sigma_{\text{TOM}}^2) \), non-i.i.d. nature of real estate returns \( (\beta) \), and transaction cost \( (C) \) affect the optimal allocation to real estate in the mixed-asset portfolio. It is perhaps necessary to note that NCREIF data is for demonstration only. The model developed in the previous section is applicable to any thinly-traded assets whose transaction process can be described as the sequential search process depicted in Figure 2.\(^{10}\)

To obtain a closed form formula for \( w^*_\text{RE} \) is difficult. However, we can find a numerical solution if we have all the model parameters at hand: (1) real estate illiquidity, i.e., \( t_{\text{TOM}} \) and \( \sigma_{\text{TOM}}^2 \); (2) the periodic return and risk of real estate asset, \( u_{\text{RE}} \) and \( \sigma_{\text{L}} \); (3) the real estate transaction cost \( (C) \); (4) the slope of the risk curves \( (\beta) \); (5) the periodic return and risk of the financial asset, \( u_{\text{L}} \) and \( \sigma_{\text{L}} \); (6) the correlation between real estate and financial asset \( (\rho) \); (7) the risk aversion parameter \( (\lambda) \) and the risk-free rate \( (r_f) \). We discuss these parameters in turn.

(1) Real estate illiquidity. Based on the information from the National Association of Realtors, the average TOM for the US residential market during the past three decades was about 7.5 months. Considering the fact that average TOMs in commercial markets are often longer than those in residential markets, we thus choose the average TOM of 8 months for demonstration purpose. In addition, we choose a range of \( \sigma_{\text{TOM}} \) from 2 months to 8 months to see how the uncertainty of TOM affects the optimal allocation to real estate.

\(^{10}\) Interested readers may see Tirole (2011) for a comprehensive discussion on the characteristics of various illiquid assets.
(2) The periodic return \((u_{RE})\) and risk \((\sigma_{RE})\) of the real estate. Using the NCREIF property index during 1978Q1 – 2007Q2, we obtain an average annual return of 10.29% and a standard deviation of 6.34%, and use them as estimate for \(u_{RE}\) and \(\sigma_{RE}\). For demonstration purpose, we make no attempt to correct any smoothing bias in the NCREIF data. Although smoothing is a known issue with the NCREIF index, recent studies have found that the effect of smoothing is trivial and may actually overstate the true volatility.\(^{11}\)

(3) The real estate transaction cost. According to Collett, Lizieri and Ward (2003), the “round-trip lump-sum costs” of real estate can be approximately 7 to 8 percent of the value of an asset. In order to see how transaction cost affects the optimal holding period and real estate allocation, we choose a range of \(C\) from 6% to 10%.

(4) The slope of the risk curves \((\beta)\). According to Cheng, Lin, and Liu (2010b), \(\beta\) for the NCREIF Property Indices is in the range of 0.82 – 0.96. Therefore, we consider three scenarios of \(\beta = 0.8, 0.9,\) and 1.0.

(5) The periodic return \((u_L)\) and risk \((\sigma_L)\) of the financial asset. Based on the S&P 500 index during the corresponding period of 1978Q1 to 2007Q2, we obtain an average annual return of 10.0% and the standard deviation of 17.5% for \(u_L\) and \(\sigma_L\).

(6) The correlation between real estate and financial asset \((\rho)\). It has been well documented in the literature that real estate historically has low correlation with financial assets. For example, using annual U.S. data from 1947 to 1982, Ibbotson and Siegel (1984) found real estate's correlation with S&P stocks to be -0.06, Worzala and Vandell (1993) estimate the

\(^{11}\) See, for example, Graff and Young (1999) and Cheng, Lin, and Liu (2011).
correlation between NCREIF quarterly index from 1980 to 1991 and stocks of the same period to be about -0.0971. Eichholtz and Hartzell (1996) document correlations between real estate and stock indexes to be -0.08 for U.S., -0.10 for Canada, and -0.09 for U.K. Quan and Titman (1999) examined the correlation for 17 countries and, unlike earlier studies, they find a generally positive correlation pattern in most countries, but for the U.S. such positive correlation is insignificant.

Using the quarterly NCREIF and S&P500 Indices during the period of 1978Q1 to 2007Q2, we compute the correlations over different investment horizons and the results are displayed in Table 1.12

Table 1. Long-run Sample Correlations between NCREIF and S&P500

<table>
<thead>
<tr>
<th>Holding Period (years)</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>NCREIF Overall vs. S&amp;P500</td>
<td>0.116</td>
<td>0.114</td>
<td>0.074</td>
<td>0.028</td>
</tr>
<tr>
<td>NCREIF Apartment vs. S&amp;P500</td>
<td>0.094</td>
<td>0.078</td>
<td>0.069</td>
<td>0.074</td>
</tr>
<tr>
<td>NCREIF Industrial vs. S&amp;P500</td>
<td>0.168</td>
<td>0.118</td>
<td>0.107</td>
<td>0.104</td>
</tr>
<tr>
<td>NCREIF Office vs. S&amp;P500</td>
<td>0.149</td>
<td>0.165</td>
<td>0.142</td>
<td>0.115</td>
</tr>
<tr>
<td>NCREIF Retail vs. S&amp;P500</td>
<td>-0.008</td>
<td>-0.079</td>
<td>-0.117</td>
<td>-0.107</td>
</tr>
</tbody>
</table>

Table 1 indicates that the magnitudes of the correlations are quite consistent with previous studies in that all coefficients are rather small. There is somewhat a trend that the correlations seem to slightly decline as holding period increases. This result, along with the findings by previous studies, reaffirms that real estate has low correlations with financial assets. Therefore, we choose a range of the correlation from -10% to 10% for our numerical analysis.

12 To compute the correlations, we first obtain the holding-period returns of various NCREIF indices and S&P 500 by replicating the method illustrated in Appendix I. Then for each holding-period, we compute the Spearman correlation coefficients between NCREIF indices and S&P 500.
The risk aversion parameter ($\lambda$) and the risk-free rate ($r_f$). The presence of a risk-aversion parameter in the model suggests that optimal asset allocation is investor specific. However, although the market is full of investors with various degrees of risk-aversion, the competitive force of the marketplace ensures that only the highest bidder gets the deal, and these highest bidders are likely to be investors who have a particular degree of risk-aversion such that the property’s expected risk-adjusted return is the highest to them. The degree of risk-aversion of those highest bidding investors, therefore, can be implied from the observed market data. As mentioned before, studies in the past using the classical MPT have reported real estate allocations to be between 15% and 40%. If we take the middle point of this range, 27% for instance, it will imply a risk-aversion parameter $\lambda = 31.3$, according to Equation (12) with $\rho = 0$. In addition, we choose $r_f = 3.5\%$ based on the estimation of risk-free rate for the corresponding period.

For holding period $T$, a number of empirical studies have shed light on the issue of typical real estate holding period. For example, Webb (1984) and Webb and McIntosh (1986) find that most real estate investors expect to hold their properties for ten years or less. Gau and Wang (1994) analyze over 1,000 Canadian commercial real estate transactions and find an average holding periods of about five to eight years, depending on property types. Fisher and Young (2000) find the median holding period for properties in the NCREIF database to be about 11 years. Collett et al. (2003) study the U.K. property market (where no depreciation is allowed for commercial properties) and find that institutional property holding period changes over time and varies by property type. They find that the median holding period of UK properties generally

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13 Numerous academic studies since the 1980s often conclude that real estate should constitute 15% to 40% or more of a diversified portfolio. See, for example, Hartzell, Hekman and Miles (1986), Fogler (1984), Firstenberg, Ross and Zisler (1988), and Hudson-Wilson, Fabozzi and Gordon (2005), among others.
fell from around 12 years in the early 1980s to less than 8 years in the late 1990s. Through a sample of small apartment buildings over the period from 1970 to 1990 in the city of San Diego, Brown and Geurts (2005) find that the average holding period for these properties is around 5 years. Overall, these studies seem to suggest a range of holding period of 5 – 11 years. For demonstration purpose, we simply take the middle point of this range by assuming an investor with a holding period of 8 years.

With all the parameters in hand, we conduct a quadratic programing procedure to solve the optimization problem of (24) and obtain the optimal weight for real estate, shown in Table 2.

Table 2. Optimal Weight for Real Estate in a Mixed-asset Portfolio

<table>
<thead>
<tr>
<th>Transaction Cost (C)</th>
<th>$\beta = 1.0$</th>
<th>$\beta = 0.9$</th>
<th>$\beta = 0.8$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2</td>
<td>5</td>
<td>8</td>
</tr>
<tr>
<td>10%</td>
<td>4.28%</td>
<td>4.25%</td>
<td>4.20%</td>
</tr>
<tr>
<td>5%</td>
<td>4.49%</td>
<td>4.46%</td>
<td>4.41%</td>
</tr>
<tr>
<td>10%</td>
<td>4.73%</td>
<td>4.69%</td>
<td>4.63%</td>
</tr>
<tr>
<td>-5%</td>
<td>4.98%</td>
<td>4.94%</td>
<td>4.88%</td>
</tr>
<tr>
<td>-10%</td>
<td>5.25%</td>
<td>5.22%</td>
<td>5.15%</td>
</tr>
<tr>
<td>10%</td>
<td>4.39%</td>
<td>4.36%</td>
<td>4.30%</td>
</tr>
<tr>
<td>5%</td>
<td>4.60%</td>
<td>4.57%</td>
<td>4.51%</td>
</tr>
<tr>
<td>8%</td>
<td>4.83%</td>
<td>4.80%</td>
<td>4.74%</td>
</tr>
<tr>
<td>-5%</td>
<td>5.08%</td>
<td>5.05%</td>
<td>4.98%</td>
</tr>
<tr>
<td>-10%</td>
<td>5.36%</td>
<td>5.32%</td>
<td>5.25%</td>
</tr>
<tr>
<td>10%</td>
<td>4.49%</td>
<td>4.46%</td>
<td>4.40%</td>
</tr>
<tr>
<td>5%</td>
<td>4.71%</td>
<td>4.67%</td>
<td>4.61%</td>
</tr>
<tr>
<td>6%</td>
<td>4.94%</td>
<td>4.90%</td>
<td>4.84%</td>
</tr>
<tr>
<td>-5%</td>
<td>5.19%</td>
<td>5.15%</td>
<td>5.09%</td>
</tr>
<tr>
<td>-10%</td>
<td>5.47%</td>
<td>5.43%</td>
<td>5.36%</td>
</tr>
</tbody>
</table>
Table 2 displays the simulated range of the optimal real estate weight under various combinations of four parameters - $\beta$, $\rho$, $\sigma_{TOM}$, and $C$. Three observations can be made from the nine panels within Table 2.

First, despite so many parameters of wide ranges, the overall range of the optimal weight of real estate seems fairly narrow and stable between 4.20% and 8.39%. Compared with previous findings based on the classical MPT, these real estate allocations are much more in line with the reality of institutional portfolios.

Second, within each panel, it can be seen that both real estate illiquidity and its correlation with the financial asset affect the optimal weight to real estate. Consistently, higher liquidity risk (higher $\sigma_{TOM}$) leads to lower allocation. Conversely, lower correlation leads to higher allocation. Closer inspection of the weights within each panel reveals that the correlation has slightly bigger impact on the optimal weight than liquidity risk. This is perhaps because at holding period of 8 years, the impact of liquidity risk is largely mitigated by the relatively long holding period.

Third, given a set of $\rho$ and $\sigma_{TOM}$, across the three panels corresponding to a given transaction cost, say, $C = 10\%$, the non-i.i.d. nature of real estate returns ($\beta$) has significant impact on the optimal weight to real estate. The higher the $\beta$, the lower the real estate allocation. A slight decline of $\beta$ from 1.0 to 0.8 is enough to increase the real estate allocation by about 2 – 3 percentage points. On the other hand, the impact of transaction cost seems relatively small when all other things being equal. This is perhaps because that at 8 years of holding period, the transaction cost is effectively amortized and its impact becomes marginalized. Overall, these
results suggest that the alternative model is able to provide a viable solution to the decades-old real estate allocation puzzle.

It may be of interest to compare our approach and results with a few recent studies that have attempted at resolving the real estate allocation puzzle. MacKinnon and Zaman (2009) examine the effect of long investment horizon on optimal real estate allocation using the VAR model developed in Campbell and Viceira (2005). Applying the model to U.S. real estate data, they find that, although real estate does exhibit long-run mean-reversion, the behavior is weaker than equities, which implies that real estate is almost as risky as equities in the long-run. But despite of this, they find that the optimal allocations to real estate remains rather high (17-31%) and has the tendency to be higher as the investment horizon increases. Rehring (2011) adopt the same Campbell-Viceira method to examine U.K. commercial property data. Unlike MacKinnon and Zaman (2009), he finds that the model-predicted real estate returns to exhibit decreasing variance in the longer horizon, which is consistent with his finding that the optimal real estate allocation increases rather rapidly as holding period increases. Therefore, neither study is able to resolve the real estate allocation puzzle. In contrast, using a somewhat different auto-regression model, Pagliari (2011) finds that, although real estate variance “decays” over the long-run, the variance of financial asset decays faster, thus making real estate relatively more risky as holding period gets longer. This finding, accompanied by what he finds to be increased correlation between private and public assets in the long run, causes real estate to be less appealing and thus carry less weight in mixed-asset portfolios. Pagliari suggests about 12% allocation in real estate for investors with a four-year investment horizon, which is still much higher than the reported 3-5% institutional reality. None of these three studies addresses real estate liquidity risk. In an effort to incorporate illiquidity into mixed-asset portfolio decision, Anglin and Gao (2011)
develops a model to discuss the impact of liquidity and liquidity shock on portfolio. But they did not provide what the optimal real estate allocation should be.

The differences between our approach and these afore-mentioned studies are obvious. First, the approaches by MacKinnon and Zaman (2009), Rehring (2011) and Pagliari (2011) essentially remain within the realm of empirically modifying the way MPT is applied to real estate, that is, they attempt to “fine-tune” the way we use MPT, not MPT itself. Our approach, on the other hand, is to modify the theory by extending it to explicitly accommodate the unique real estate features based on multi-period utility maximization. Second, while the previous studies attempt at obtaining better empirical estimates for the input data (long-run mean and variances) to MPT, we focus on developing an alternative portfolio model.

6. Conclusions

Modern Portfolio Theory (MPT) is a single-period asset allocation model. Its validity on multi-period portfolio decisions hinges on a critical assumption – an efficient market where asset returns are independent and identically distributed (i.i.d.) over time. Because real estate does not fit in this paradigm, the classical MPT needs to be extended to accommodate the more complex features of real estate and mixed-asset portfolio analysis. Building upon a series of recent research on real estate illiquidity and performance metrics, this paper synthesizes some of the latest advances in the literature to develop an alternative model that extends the classical MPT for mixed-asset portfolio analysis. Unlike many previous efforts that attempt to empirically solve the real estate allocation puzzle with *ad hoc* solutions, we provide a formal model that explicitly incorporates the three most unique features of real estate – horizon-dependent performance, liquidity risk, and high transaction cost – into a multi-period mean-variance analysis. Using
commercial real estate data, the alternative model produces a range of optimal real estate allocations that are quite in line with the reality of institutional portfolios.

It should be acknowledged that, although our model is able to succeed where the conventional approach has failed in resolving the long-standing real estate allocation puzzle, this work should only be viewed as a first step in the search for a more general portfolio theory for mixed-asset portfolio analysis. Many issues still remain. To mention a few: First, we have implicitly assumed that the investors are “normal” sellers who are not under any liquidity shock to force liquidation of real estate. A more general theory should incorporate seller heterogeneity and the possibility of liquidity shock into the model. Second, we do not consider the indivisibility or partial sale of real estate asset. We assume the entire real estate holding will be sold together. In reality, investors facing liquidity shock may need to liquidate only part of their real estate holdings to satisfy the need for cash. This issue is discussed in Anglin and Gao (2011) to some extent. Third, while our alternative model breaks away from the efficient market paradigm in which asset prices are assumed to follow the random walk, it is still confined within the mean-variance framework, which implicitly assumes that investors have a symmetric aversion to price volatility. The reality, though, is that investor’s risk perception is often asymmetric. To the extent that asset returns are not jointly normally-distributed, the concept of downside-risk is an appealing risk metric (and more complex, too). This, of course, is a much broader issue that pertains to nearly all mainstream finance theories as well. Despite these simplifications, the empirical analysis and theoretical exploration accomplish the two modest objectives of this paper—to extend the MPT for mixed-asset portfolio analysis and to suggest a solution to the decades-old real estate allocation puzzle. While this line of work is certainly at the stage of early exploration, the prospect is promising.
References


Appendix I.

The table below illustrates the computation of the risk curves in Figure 1 using the NCREIF Index (national). The method is replicated from Cheng, Lin, Liu (2010b). Due to the space limit, only selected holding periods are displayed.

<table>
<thead>
<tr>
<th>Year</th>
<th>NCREIF U.S. (All Property)</th>
<th>NCREIF U.S. Indexed</th>
<th>Holding Periods (Quarters)</th>
</tr>
</thead>
<tbody>
<tr>
<td>197803</td>
<td>2.90%</td>
<td>102.90</td>
<td>2.90%</td>
</tr>
<tr>
<td>197802</td>
<td>3.07%</td>
<td>106.06</td>
<td>3.07%</td>
</tr>
<tr>
<td>197803</td>
<td>3.39%</td>
<td>109.65</td>
<td>3.39%</td>
</tr>
</tbody>
</table>

Holding-period return calculation:
For a given holding period, e.g. 5 years or 20 qtrs, "buy" the Index in any quarter and "sell" 20 qtr later and record the return.

Example: "Buy" Index at D5, "Sell" Index at D25, 5-year Return = D25/D5 - 1

<table>
<thead>
<tr>
<th>Year</th>
<th>NCREIF U.S. (All Property)</th>
<th>NCREIF U.S. Indexed</th>
<th>Holding Periods (Quarters)</th>
</tr>
</thead>
<tbody>
<tr>
<td>197803</td>
<td>2.90%</td>
<td>102.90</td>
<td>2.90%</td>
</tr>
<tr>
<td>197802</td>
<td>3.07%</td>
<td>106.06</td>
<td>3.07%</td>
</tr>
<tr>
<td>197803</td>
<td>3.39%</td>
<td>109.65</td>
<td>3.39%</td>
</tr>
</tbody>
</table>

The Standardized St. Dev is plotted in the Figure 1 as the NCREIF All Property line. The same computation is replicated for all other lines.

Mean Returns: 2.48% 10.29% 21.37% 32.79% 44.57% 56.96% 70.49% 84.66% 99.07% 113.72%

St. Dev. 1.70% 6.34% 12.90% 19.12% 24.93% 30.80% 37.70% 44.80% 51.86% 59.15%

Standardized St. Dev. 1.00 3.73 7.59 11.26 14.68 18.13 22.20 26.38 30.53 34.82
Some readers may have concerns with the calculation above using overlapping periods. We tried to use non-overlapping periods for the calculation and find that the results are not much different but the method raises other issues. First, given the length of the NCREIF index, it is unlikely to obtain meaningful results for long holding periods. For example, for holding period of 36-quarter (9 years), we only have three non-overlapping periods. Second, using non-overlapping periods creates a loss of information, as many data points are not used, which effectively means that we would use only a small part of the sample instead of the whole sample for the calculation, especially for longer holding periods. Third, perhaps the most serious problem is that using non-overlapping periods introduces seasonality bias into the calculation. For example, if the index happens to begin on the first quarter of 1978, a 4-quarter holding-period return series with non-overlapping data would not use index points in the other 3 quarters, which implies that investors can only buy and sell in the first quarter of every year (or every other year if the holding period is 8 quarters, and so on). We think such loss of information is greater than the problem of calculation with overlapping periods. So we have tried an alternative method for comparison. For the 4-quarter holding period, for instance, we subdivided the whole series by the 4 quarters so that each subgroup only contains data from the same quarter of every year. We first calculate the mean and variance for each subgroup, then average them across the subgroups. Because there is no overlapping data within each subgroup, this method should be superior in principle to the method with overlapping periods, but it requires much more computational work. The results based on the two methods were close; the means are identical and the variances are similar across different holding periods.