Pohozaev's identity and overdetermined problems in partial differential equations

B. Emamizadeh
School of Mathematical Sciences
E-mail: Behrouz.Emamizadeh@nottingham.edu.cn

I intend to give a series of lectures, during the upcoming break, on overdetermined problems. I hope these lectures will lead to writing an expository paper on this topic. I am not an expert in this area but an enthusiast. I believe I have published at least two papers in this area. I don't remember when I wrote the first one but the second one will appear in 2021. Although it is a fascinating area of research, it is very challenging to find a good problem to work on and even more challenging to find techniques to plow through obstacles on your way.

Let me now introduce you to the most well-known and basic overdetermined problem in partial differential equations. The boundary value problem

$$(TP) \qquad \begin{cases} -\Delta u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

is known as the *torsion problem* or the *Saint-Venant problem*. Here Ω, the domain of interest, is a bounded region in $\mathbb{R}^n$, and has smooth boundary $\partial\Omega$. The unknown u is a function of the spatial variables $x_1, \cdots, x_n$ i.e. $u = u(x_1, \cdots, x_n)$. The notation Δu denotes the Laplacian of u:

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2}.$$

In (TP) one seeks a function u with Laplacian equal to -1 throughout Ω, and vanishing on $\partial\Omega$. In particular domains u can be explicitly found. One such domain is an n-dimensional ball, say, centered at $x^0 \in \mathbb{R}^n$ with radius R, denoted $\mathcal{B}(x^0, R)$. One easily verifies that the problem

$$\begin{cases} -\Delta u = 1 & \text{in } \mathcal{B}(x^0, R) \\ u = 0 & \text{on } \partial\mathcal{B}(x^0, R) \end{cases}$$

has the solution, $u(x) = \frac{R^2 - |x-x^0|^2}{2n}$. Here $x = (x_1, \cdots, x_n)$ and $|x - x^0|^2 = \sum_{k=1}^n (x_k - x_k^0)^2$. One interesting fact about this solution is that its *normal derivative* on $\partial\mathcal{B}(x^0, R)$ is constant. Indeed, if we denote the unit normal vector on $\partial\mathcal{B}(x^0, R)$ pointing out of $\mathcal{B}(x^0, R)$ by ν, then

$$\frac{\partial u}{\partial \nu}|_{\partial\mathcal{B}(x^0,R)} = -\frac{R}{n}.$$

A question is then raised: Is the converse of this result true? Let us state this as a
Conjecture: Suppose the system

$$(OD) \qquad \begin{cases} -\Delta u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ \frac{\partial u}{\partial \nu} = \text{constant} =: c & \text{on } \partial\Omega \end{cases}$$

has a solution u. Then D must be an n-ball.

In the literature a problem like (OD) is called *overdetermined*, why? The reason is obvious. The problem (TP) is already well-posed; meaning, it has a unique solution. Hence the condition:

$$\frac{\partial u}{\partial \nu} = c \quad \text{on } \partial\Omega$$

is simply *extra*; that is the reason for using the word "overdetermined". I used to call it the "annoying" condition but not anymore. I am convinced that overdetermined problems are mathematically fascinating, and they are indeed the main drive behind the birth of many powerful mathematical techniques.

In 1971, exactly half a century ago, James Serrin settled the conjecture affirmatively, see [1]. Interestingly, the very next paper in the same journal was authored by Hans Weinberger, in which he also gave an affirmative answer to the conjecture, but his approach was completely different than that of Serrin's, see [2]. I had the privilege to meet James in 1997 in the UK, and (only) see Hans (several times) during the Bieberbach conjecture conference held at Purdue in 1985. Serrin's approach used the method of the *moving plane* while Weinberger's, more or less, used *advanced calculus*. The latter was a lot more appealing to me. Over years I have been returning to Weinberger's paper on different occasions and for different purposes either because I had to or I just wanted it for entertainment. On one of these visits I noticed an important identity in his proof:

$$\int_\Omega u \, dx = \gamma(c, n, V),$$

with an explicit formulation of γ. Here, V stands for the volume of Ω, c is the constant appearing in the second boundary condition, and n is the dimension. This identity was the main key in his proof. What puzzled me was that he worked unnecessarily too hard to obtain the identity when he could readily get it by applying the *Pohozaev's identity*:

Theorem 0.1. [3] *Suppose u satisfies*

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Then the following identity holds:

$$\frac{2-n}{2}\int_\Omega |\nabla u|^2 \, dx + n\int_\Omega F(u) \, dx = \frac{1}{2}\int_{\partial\Omega} (x\cdot\nu)\, |\nabla u|^2 \, d\sigma,$$

where $F(t) = \int_0^t f(s)\, ds$.

The first version of the paper was in Russian, but it was translated into English in the same year (1965, six years before Weinberger's proof!!). The total number of citations of Pohozaev's remarkable paper to date is 371. Only 9 of these citations belong to 1965-2000. That means this paper went nearly unnoticed for 35 years. (why?). Were the Russian and US mathematical communities so much disconnected then? Whatever the reason was it is past; the international mathematical community now agrees on the profound importance of Pohozaev's identity, and many mathematicians are either generalizing it or using it to prove existence or non-existence of solutions to partial differential equations.

Let me write a little bit about the lectures. I will use four differential operators throughout. They are

• The Laplace operator Δ;
• The p-Laplace operator Δ_p. The action of Δ_p on a function $u(x_1, \cdots, x_n)$ is defined as follows:

$$\Delta_p u(x) = \nabla \cdot (|\nabla u(x)|^{p-2} \nabla u(x)),$$

where $p \in [1, \infty)$. The symbol ∇ denotes the gradient operator i.e. $\nabla u(x) = \langle \partial u/\partial x_1, \cdots, \partial u/\partial x_n \rangle$. Actually, when $p = \infty$, we have the infinity-Laplace operator but I will not use this operator because the solutions of differential equations involving Δ_∞ are of *viscosity* type and this topic is irrelevant to the main theme of the lectures;
• The bi-Laplace operator $\Delta^2 := \Delta(\Delta)$;
• The k-Hessian operators S_k. These operators are probably not familiar to many people. So let me write a bit about them. Let A be a symmetric real square matrix of size $n \in \mathbb{N}$. It is well known, and of course easy to prove, that the eigenvalues of A are real. Given the size of A is n, there must be n eigenvalues, counting multiplicities. Let us denote them by $\lambda_1, \cdots, \lambda_n$. For $k \in \{1, \cdots, n\}$, the k-th elementary symmetric function of the eigenvalues of A is defined as follows:

$$S_k(A) = \sum_{i_1 < i_2 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}.$$

So, for example, when $n = 4$ and $k = 2$, we have

$$S_2(A) = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4.$$

Said another way, $S_k(A)$ is the sum of all k-dimensional principle minors of A. How do we construct a differential operator out of S_k? We begin with $u = u(x_1, \cdots, x_n)$, say a smooth function. The *Hessian matrix* of u is the square matrix:

$$\mathcal{H}(u)(x) = \begin{pmatrix} \partial^2 u/\partial x_1^2 & \partial^2 u/\partial x_1 \partial x_2 & \cdots & \partial^2 u/\partial x_1 \partial x_n \\ \vdots & \vdots & \ddots & \vdots \\ \partial^2 u/\partial x_n \partial x_1 & \partial^2 u/\partial x_n \partial x_2 & \cdots & \partial^2 u/\partial x_n^2 \end{pmatrix}$$

I shall abuse the notation and write $\nabla^2 u$ instead of $\mathcal{H}(u)$. Since $\nabla^2 u$ is a symmetric matrix it makes sense to talk about $S_k(\nabla^2 u)$, the so called k-Hessian differential operator. Needless to say, the eigenvalues of $\nabla^2 u$ depend on the spatial variable x. Usually we hear of linear or nonlinear differential operators but, for your information, $S_k(\nabla^2 u)$ is classified as a *fully* nonlinear differential operator, whatever that means. To get a feel of $S_k(\nabla^2 u)$, let us look at two very special cases:
i) $k = 1$. In this case, $S_1(A) = \lambda_1 + \cdots + \lambda_n$. That is to say, $S_k(A) = \mathrm{trace}(A)$. Whence, $S_1(\nabla^2 u) = \mathrm{trace}(\nabla^2 u) = \Delta u$. In other words, S_1 coincides with the Laplace operator;

ii) $k = n$. In this case, $S_n(A) = \lambda_1 \cdots \lambda_n$. Hence, $S_n(A) = \det(A)$. So, $S_n(\nabla^2 u) = \det(\nabla^2 u)$, which is known as the Monge-Ampère operator. Amongst other things I will derive the following Pohozaev's identity:

Theorem 0.2. *Let $f \in C^1(\Omega)$ be a non-negative function and $F(u) = \int_u^0 f(s)\, ds$. If $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies:*

$$\begin{cases} S_k(\nabla^2 u) = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded C^2 domain in $\mathbb{R}^n$, then

$$\frac{n-2k}{k(k+1)} \int_\Omega S_k^{ij}(\nabla^2 u) u_i u_j \, dx + \frac{1}{k+1} \int_{\partial\Omega} \langle x, \nu \rangle \, \nabla u|^{k+1} \, d\sigma = n \int_\Omega F(u)\, dx,$$

where, in general, $S_k^{ij}(A) = \frac{\partial S_k(A)}{\partial a_{ij}}$.

References

[1] Serrin, J., A symmetry problem in potential theory. Arch. Rational Mech. Anal. 43, (1971), 304−318.
[2] Weinberger, H. F. Remark on the preceding paper of Serrin. Arch. Rational Mech. Anal. 43, (1971), 319−320.
[3] Pohoaev, S. I. On the eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$. (Russian) Dokl. Akad. Nauk SSSR 165, (1965), 36−39.
[4] Willms, N. B.; Chamberland, Marc; Gladwell, G. M. L. A duality theorem for an overdetermined eigenvalue problem. Z. Angew. Math. Phys. 46, (1995), no. 4, 623−629.
[5] Brandolini, B.; Nitsch, C.; Salani, P.; Trombetti, C. Serrin-type overdetermined problems: an alternative proof. Arch. Ration. Mech. Anal. 190, (2008), no. 2, 267−280.
[6] Bennett, Allan, Symmetry in an overdetermined fourth order elliptic boundary value problem. SIAM J. Math. Anal. 17, (1986), no. 6, 1354−1358.