Modified scattering operator for nonlinear Schrödinger equations with a time-decaying harmonic potential

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We consider the nonlinear Schrödinger equation

$$\text{(NLS)} \qquad i\partial_t u - H(t)u = |u|^{p_c}u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d,$$

where $d \leqslant 3, \quad p_c = \frac{2}{d(1-\lambda)}, \quad \lambda = \lambda(\sigma_1) \in [0, 1/2),$

$$H(t) = -\frac{\Delta}{2} + \sigma(t)\frac{|x|^2}{2}, \quad \sigma(t) = \begin{cases} \sigma_0 & (|t| < r_0), \\ \sigma_1 |t|^{-2} & (|t| \geqslant r_0), \end{cases} \quad \sigma_j,\ r_0 > 0.$$

NOTE: We can handle $\sigma(t)$ in more general, and the case $\lambda < 0$.

Our Aim: To construct a modified scattering operator S_+.

Final State Problem For given u_p^-, find $u(t) \to u_p^-(t)$ as $t \to -\infty$.

Initial Value Problem For given u_0, find $u(t) \to u_p^+(t)$ as $t \to +\infty$.

Critical exponent $p_c = 2/d$ in the case $\sigma(t) \equiv 0$

(NLS$_0$) $\qquad i\partial_t u + \frac{1}{2}\Delta u = |u|^p u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d.$

(NLS$_0$) & $u(t) \sim e^{it\Delta/2}u_\pm$ as $t \to \pm\infty$ (given $u_\pm$: final state)

$$\iff u(t) = e^{it\Delta/2}u_\pm \mp i \int_t^{\pm\infty} e^{i(t-s)\Delta/2}|u(s)|^p u(s)\, ds.$$

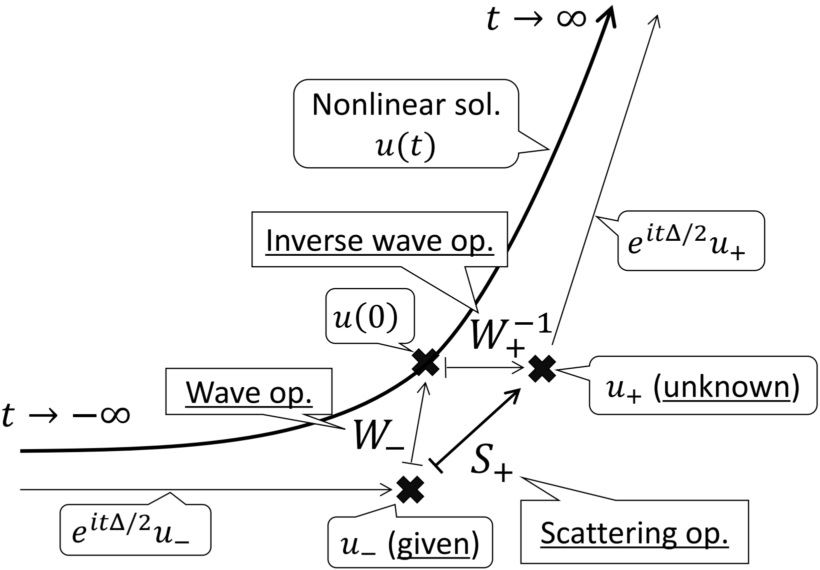
NOTE: $\left\|e^{it\Delta/2}u_\pm\right\|_{L^\infty} \lesssim |t|^{-\frac{d}{2}}, \quad \left\|e^{it\Delta/2}u_\pm\right\|_{L^2} = \|u_+\|_{L^2}.$

If $u(t) \sim e^{it\Delta/2}u_\pm$, then

$$\left\| \int_t^{\pm\infty} e^{i(t-s)\Delta/2}|e^{is\Delta/2}u_\pm|^p e^{is\Delta/2}u_\pm\, ds \right\|_{L^2} \lesssim \int_t^{\pm\infty} |s|^{-\frac{d}{2}p}\, ds.$$

It is expected that $u(t) \to e^{it\Delta/2}u_\pm$ if $-dp/2 < -1 \Leftrightarrow p > 2/d$.

$p > 2/d$: $u(t) \to e^{it\Delta/2}u_\pm$. Scattering Tsutsumi-Yajima '84.



$p \leqslant 2/d$: $\nexists u(t) \to e^{it\Delta/2}u_\pm$. Failure of scattering Barab '84.

$p = 2/d$: $u(t) \to \mathcal{M}(t)\mathcal{D}(t) \exp\left(-i \left|\widehat{u_\pm}\right|^{\frac{2}{d}} \log|t|\right) \widehat{u_\pm}$,

where $\mathcal{M}(t) = e^{\frac{i|x|^2}{2t}}$, $[\mathcal{D}(t)f](x) = (it)^{-\frac{d}{2}} f(x/t)$.

Modified scattering NOTE: $\mathcal{M}(t)\mathcal{D}(t)\widehat{\phi} \sim e^{it\Delta/2}\phi$ as $|t| \to \infty$.

(FS) Ozawa '91, $d = 1$, Ginibre-Ozawa '93, $d = 2, 3$,

(IV) Hayashi-Naumkin '98.

In what follows, we only focus on the case $p = p_c$.

Modified scattering operator

Set $d/2 < \delta < \beta < \alpha < \min(1 + p_c, d), \quad m \in \mathbb{R}$,

$$B_\varepsilon^\alpha = \{\phi \in H^{0,\alpha} \mid \|\phi\|_{H^{0,\alpha}} < \varepsilon\}, \ \|f\|_{H^{0,m}} = \left\|(1 + |x|^2)^{m/2} f\right\|_{L^2}.$$

Hayashi-Naumkin '06: $\varepsilon > 0$: small, $c_0 > 0$: const.

$$\exists W_- \colon B_\varepsilon^\alpha \ni u_- \mapsto u(0) \in B_{c_0\varepsilon}^\beta, \quad \exists W_+^{-1} \colon B_\varepsilon^\beta \ni u(0) \mapsto u_+ \in H^{0,\delta},$$

$$\mathrm{Ran}(W_-) \subset \mathrm{Dom}(W_+^{-1}), \ \exists S_+ := W_+^{-1} W_- \colon B_\varepsilon^\alpha \ni u_- \mapsto u_+ \in H^{0,\delta}.$$

Theorem 1 (Modified scattering operator for (NLS))

$$\exists S_+ := W_+^{-1} W_- \colon B_\varepsilon^\alpha \ni u_- \mapsto u_+ \in H^{0,\delta}.$$

NOTE: The formulation is a bit different from that of H-N '06.

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Case $\sigma(t) = \sigma_1 |t|^{-2}$ $(|t| \geq r_0)$, σ_0 $(|t| < r_0)$

Consider the initial value problem for

$$(1) \qquad \begin{cases} \zeta_j''(t) + \sigma(t)\zeta_j(t) = 0, \quad j = 1,\, 2, \\ \zeta_1(0) = 1,\ \zeta_1'(0) = 0,\ \zeta_2(0) = 0,\ \zeta_2'(0) = 1, \end{cases}$$

where $\sigma_1 \in [0, 1/4)$, σ_0, $r_0 > 0$: suitable constant. $|t|^{\lambda}$, $|t|^{1-\lambda}$ solve

$$y''(t) + \sigma(t)y(t) = 0 \quad (|t| \geqslant r_0), \quad \lambda := \frac{1 - \sqrt{1 - 4\sigma_1}}{2} \in \left[0, \frac{1}{2}\right).$$

Then $\exists \zeta_j \in C^2$: sol. to (1) s.t. for some $c_i \in \mathbb{R}$ with $c_4 \neq 0$,

$$\zeta_1(t) = \begin{cases} \cos\left(\sqrt{\sigma_0}|t|\right), \\ c_1|t|^{\lambda} + c_2|t|^{1-\lambda}, \end{cases} \quad \zeta_2(t) = \begin{cases} \frac{1}{\sqrt{\sigma_0}} \sin\left(\sqrt{\sigma_0}|t|\right) & (|t| < r_0), \\ c_3|t|^{\lambda} + c_4|t|^{1-\lambda} & (|t| \geqslant r_0). \end{cases}$$

$U(t,s)$: propagator of $H(t) := -\frac{\Delta}{2} + \sigma(t)\frac{|x|^2}{2}$, $\sigma(t) \sim |t|^{-2}$

cf. Fujiwara '79.

- Quantum particle governed by $H(t)$:

  $\sigma(t) \sim |t|^{-\rho}$: $\rho < 2$: trapped; $\rho > 2$: scatter.

- The particle is decelerated as $|t| \to \infty$, but not trapped:

  Ishida-Kawamoto '20.

$$\|U(t,0)u_+\|_{L^\infty} \lesssim |\zeta_2(t)|^{-\frac{d}{2}} \lesssim |t|^{-\frac{d}{2}(1-\lambda)}. \quad p_c = \frac{2}{d} \to \frac{2}{d(1-\lambda)}.$$

Lemma 1 (Korotyaev '90, Kawamoto-Muramatsu '21)

$$U(t,0) = \mathcal{M}\left(\frac{\zeta_2(t)}{\zeta_2'(t)}\right)\mathcal{D}(\zeta_2(t))\mathcal{F}\mathcal{M}\left(\frac{\zeta_2(t)}{\zeta_1(t)}\right).$$

- $\mathcal{M}(\zeta_2(t)/\zeta_1(t)) \sim \mathcal{M}(c_4/c_2) =: \mathcal{M}_+$ as $|t| \to \infty$.

$$d/2 < \beta < \alpha < \min(1+p_c, d), \quad B^\alpha_\varepsilon = \{\phi \in H^{0,\alpha} \mid \|\phi\|_{H^{0,\alpha}} < \varepsilon\}$$

Proposition 1 (Final State Problem)

Let $u_- \in B^\alpha_\varepsilon$ with $\varepsilon \ll 1$. Then $\exists! u \in C\left((-\infty, -r_0]; L^2\right)$: sol. to (NLS) with $U(0,\cdot)u \in C\left((-\infty, -r_0]; H^{0,\beta}\right)$. Moreover

$$\forall t \leqslant -r_0, \quad \left\| U(0,t)\left(u(t) - u_p^-(t)\right)\right\|_{H^{0,\beta}} \lesssim \varepsilon |t|^{-\mu}, \quad \mu > 0\text{: small,}$$

$$u_p^-(t) := \mathcal{M}\left(\frac{\zeta_2(t)}{\zeta_2'(t)}\right)\mathcal{D}(\zeta_2(t)) \exp\left(\frac{i\,|\widehat{u_-}|^{p_c}}{c_+}\log|t|\right)\widehat{u_-}, \; c_+ := |c_4|^{\frac{1}{1-\lambda}}.$$

By the persistence of regularity of solutions and conserved $\|u(t)\|_{L^2}$,

$$\exists W_-\colon B^\alpha_\varepsilon \ni u_- \mapsto u(0) \in B^\beta_{c_0\varepsilon}, \quad \varepsilon > 0\text{: small, } c_0 > 0\text{: const.}$$

cf. Kawamoto '21: $\quad \exists W_-\colon B^\alpha_\varepsilon \ni u_- \mapsto u(0) \in L^2.$

$$d/2 < \delta < \beta < \min(1+p_c, d), \quad B_\varepsilon^\beta = \{\phi \in H^{0,\beta} \mid \|\phi\|_{H^{0,\beta}} < \varepsilon\}$$

Proposition 2 (Initial Value Problem)

Let $U(0,r_0)u_0 \in B_\varepsilon^\beta$ with $\varepsilon \ll 1$. Then $\exists! u \in C([r_0,\infty); L^2)$: sol. to (NLS) with $U(0,\cdot)u \in C\left([r_0,\infty); H^{0,\beta}\right)$ under $u(r_0) = u_0$. Moreover, $\exists! u_+ \in H^{0,\delta}$ s.t. $\forall t > r_0$,

$$\left\|\mathcal{F}^{-1}\mathcal{P}(t)\mathcal{F}\mathcal{M}_+U(0,t)u(t) - u_+\right\|_{H^{0,\delta}} \lesssim \varepsilon t^{-\mu}, \quad \mu > 0\text{: small},$$

where $\mathcal{P}(t) = \exp\left(i\int_{r_0}^t \psi(\tau)|\zeta_2(\tau)|^{-\frac{1}{1-\lambda}}\,d\tau\right)$, small $\chi > 0$,

$$\psi(\tau) := \begin{cases} |\mathcal{F}\mathcal{M}_+U(0,\tau)u(\tau)|^{p_c}, & \text{if } d = 1, \\ \left(|\tau|^{-\chi} + |\mathcal{F}\mathcal{M}_+U(0,\tau)u(\tau)|^2\right)^{\frac{p_c}{2}} & \text{if } d = 2, 3. \end{cases}$$

By the persistence of regularity of solutions and conserved $\|u(t)\|_{L^2}$,

$$\exists W_+^{-1}\colon B_\varepsilon^\beta \ni u(0) \mapsto u_+ \in H^{0,\delta}, \quad \varepsilon > 0\text{: small.}$$

cf. Kawamoto-Muramatsu '21: $\quad \exists W_+^{-1}\colon B_\varepsilon^{\beta,\beta} \ni u(0) \mapsto u_+ \in L^2$,

$$B_\varepsilon^{\beta,\beta} \coloneqq \{\phi \mid \|\phi\|_{H^{\beta,0}\cap H^{0,\beta}} < \varepsilon\},\ \|f\|_{H^{\beta,0}} \coloneqq \left\|(1+|\nabla|^2)^{\beta/2} f\right\|_{L^2}.$$

Since $\mathrm{Ran}(W_-) \subset \mathrm{Dom}(W_+^{-1})$ if $\varepsilon \ll 1$, Theorem 1 holds, i.e.,

$$\exists S_+ \coloneqq W_+^{-1} W_-\colon B_\varepsilon^\alpha \ni u_- \mapsto u_+ \in H^{0,\delta},$$

where $d/2 < \delta < \alpha < \min(d, 1 + p_c)$.

Remark 1

- We emphasize that our result gives an alternative construction of S_+ for (NLS) without potentials. Namely, our result includes [H-N '06], despite some formulation differences.
- In [K '21] for FS problem, the strong restriction

$$\lambda = \lambda_d \geqslant 0 : \ \lambda_1 < \frac{13-\sqrt{145}}{4}, \quad \lambda_2 < \frac{7-\sqrt{41}}{4}, \quad \lambda_3 < 0.022.$$

  is imposed, due to a use of the time-weighted Strichartz estimate. We remove the restriction by considering a solution in the space associated with $J(t) \coloneqq U(t,0)xU(0,t)$.
- In Prop 1, the same assertion holds if we replace u_- by $\mathcal{M}_+ u_-$.

Factorization of the propagator

Recall $\mathcal{M}(t) = e^{\frac{i|x|^2}{2t}}$, $\mathcal{M}_+ = \mathcal{M}\left(c_4/c_2\right)$, $[\mathcal{D}(t)f](x) = (it)^{-\frac{d}{2}} f\left(x/t\right)$. Set

$$\mathcal{M}_1(t) = \mathcal{M}\left(\frac{\zeta_2(t)}{\zeta_2'(t)}\right), \ \mathcal{D}_1(t) = \mathcal{D}(\zeta_2(t)), \ \mathcal{M}_+\mathcal{M}_2(t) = \mathcal{M}\left(\frac{\zeta_2(t)}{\zeta_1(t)}\right).$$

Then the factorization of $U(t,0)$ (Lemma 1) rewrites

Key Tool: $\quad U(t,0) = \mathcal{M}_1(t)\mathcal{D}_1(t)\mathcal{F}\mathcal{M}_+\mathcal{M}_2(t).$

We notice

$$|\mathcal{M}_2(t) - 1| \lesssim |x|^{2\theta}|t|^{-\theta(1-2\lambda)} \quad (\theta \in [0,1]).$$

Proof of Proposition 1

We follow Ozawa '91 and Hayashi-Naumkin '06. Set $F(u) = |u|^{p_c} u$,

$$u_p^-(t) = \mathcal{M}_1(t)\mathcal{D}_1(t) \exp\left(\frac{i\left|\widehat{u_-}\right|^{p_c}}{c_+} \log|t| \right) \widehat{u_-}.$$

Denote a new unknown by $v := u - u_p^-$. (NLS) rewrites

$$i\partial_t v - H(t)v = F(u_p^- + v) - F(u_p^-) - \{(i\partial_t - H(t))\, u_p^- - F(u_p^-)\},$$

$$\Longrightarrow \quad i\partial_t \left(U(0,t)v\right) = U(0,t)\{F(u_p^- + v) - F(u_p^-)\} + \mathcal{E}(t),$$

$$\text{where} \quad \mathcal{E}(t) = -i\partial_t \left(U(0,t)u_p^-\right) + U(0,t)F(u_p^-).$$

u_p^-: expected asymptotic profile. We may assume $\lim_{t\to-\infty} \|v(t)\|_{L^2} = 0$.

From the Duhamel principle, show that v solves

$$v(t) = -i \int_{-\infty}^{t} U(t,s)(F(u_p^-(s) + v(s)) - F(u_p^-(s)))\, ds + \widetilde{\mathcal{E}}(t),$$

where $\widetilde{\mathcal{E}}(t) = -i \int_{-\infty}^{t} U(t,0)\mathcal{E}(s)\, ds$, in the complete metric sp. with

$$\|\phi\|_{X_b} = \sup_{t \leqslant -r_0} \left(|t|^{\frac{\beta}{2}(1-2\lambda)+b} \|\phi(t)\|_{L^2} + |t|^b \|U(0,t)\phi(t)\|_{H^{0,\beta}} \right) \leqslant 1,$$

$$d(\phi,\psi) = \|\phi - \psi\|_{L^\infty(-\infty,-r_0;L^2(\mathbb{R}^d))}, \quad b = b(\lambda,\beta) > 0 : \text{ small.}$$

Thanks to the specific choice of u_p^-, by the factorization of $U(t,0)$,

$$\left\| U(0,t)\widetilde{\mathcal{E}}(t) \right\|_{H^{0,\beta}} = O(t^{-\frac{\alpha-\beta}{2}}),$$

since $|\mathcal{M}_2(t) - 1| \lesssim |x|^{\alpha-\beta} |t|^{-\frac{\alpha-\beta}{2}(1-2\lambda)}$.

Nonlinear estimates in $d \geqslant 2$, $d/2 < \beta < p_c + 1$

NOTE: $U(t,0)xU(0,t) = \mathcal{M}_1(t) i\zeta_2(t) \nabla \mathcal{M}_1^{-1}(t)$.

Set $\phi = \mathcal{M}_1^{-1}(t)\left(u_p^- + v\right)$ and $\psi = \mathcal{M}_1^{-1}(t) u_p$. We have

$$\left\| U(0,t) \int_{-\infty}^{t} U(t,s)(F(u_p^-(s) + v(s)) - F(u_p^-(s)))\, ds \right\|_{H^{0,\beta}}$$
$$\lesssim \int_{-\infty}^{t} |\zeta_2(s)|^{\beta} \| F(\phi(s)) - F(\psi(s)) \|_{H^{\beta}}\, ds.$$

Then $\nabla \left(F(\phi) - F(\psi)\right) \sim |\phi|^{p_c} \nabla(\phi - \psi) + \left(|\phi|^{p_c} - |\psi|^{p_c}\right) \nabla\psi$.

Key: $\phi - \psi = \mathcal{M}_1^{-1}(t) v$ appears to get necessary time decay.

However if we estimate $\left||\phi|^{p_c} - |\psi|^{p_c}\right| \lesssim \left(|\phi|^{p_c - 1} - |\psi|^{p_c - 1}\right)|\phi - \psi|$,

then we can not handle the case $p_c - 1 < \beta - 1$.

$\underline{p_c \geqslant 1}$: By fractional chain rule,

$$
\begin{aligned}
&\left\| \left(|\phi|^{p_c} - |\psi|^{p_c} \right) \nabla\psi \right\|_{\dot{H}^{\beta-1}} \\
\lesssim\ &\left\| |\phi|^{p_c} - |\psi|^{p_c} \right\|_{L^\infty} \|\psi\|_{\dot{H}^\beta} + \left\| |\nabla|^{\beta-1} \left(|\phi|^{p_c} - |\psi|^{p_c} \right) \right\|_{L^{\frac{2\beta}{\beta-1}}} \|\nabla\psi\|_{L^{2\beta}}.
\end{aligned}
$$

Note $\|\nabla\psi\|_{L^{2\beta}} \lesssim \|\psi\|_{\dot{H}^\beta}$ from Gagliardo-Nirenberg's inequality. Then

$$
\begin{aligned}
&\left\| |\nabla|^{\beta-1} \left(|\phi|^{p_c} - |\psi|^{p_c} \right) \right\|_{L^{\frac{2\beta}{\beta-1}}} \\
\lesssim\ &\left\| |\phi|^{p_c} - |\psi|^{p_c} \right\|_{L^\infty}^{2-\beta} \left\| |\nabla| \left(|\phi|^{p_c} - |\psi|^{p_c} \right) \right\|_{L^{2\beta}}^{\beta-1} \\
\lesssim\ &\left(\|\phi\|_\infty + \|\psi\|_\infty \right)^{(p_c-1)(2-\beta)} \|\phi - \psi\|_\infty^{2-\beta} \\
&\times \left(\|\phi\|_\infty^{p_c-1} \|\nabla\phi\|_{2\beta} + \|\psi\|_\infty^{p_c-1} \|\nabla\psi\|_{2\beta} \right)^{\beta-1}.
\end{aligned}
$$

$\underline{p_c < 1}$: Fractional chain rule for Hölder conti. func. cf. Vișan '07.

Proof of Proposition 2 in $d \geqslant 2$

1st Step: Show that $\exists ! u \in C([r_0, \infty); L^2)$: sol. to (NLS) with

$$|t|^{-A\varepsilon^{\frac{1}{d(1-\lambda)}}} \|U(0,t)u(t)\|_{H^{0,\beta}} + |t|^{-\frac{d}{2}(1-\lambda)} \|u(t)\|_{L^\infty} \lesssim 1.$$

$\because$) Local existence: Standard contraction argument.

Global existence: Bootstrap argument & a priori bound.

Indeed, by the integral eq. from (NLS) & Gronwall's ineq.,

$$|t|^{-A\varepsilon^{\frac{1}{d(1-\lambda)}}} \|U(0,t)u(t)\|_{H^{0,\beta}} \leqslant \varepsilon \quad (t \in [r_0, r_0 + T]).$$

By the factorization of $U(t,0)$, we see that

$$\|u(t)\|_{L^\infty} \lesssim |t|^{-\frac{d}{2}(1-\lambda)} \underbrace{\|\mathcal{F}\mathcal{M}_+ U(0,t)u(t)\|_{L^\infty}}_{\lesssim \varepsilon,\ \text{arguing as in 2nd Step.}} + o(|t|^{-\frac{d}{2}(1-\lambda)}).$$

2nd Step: Find the unique final data u_+ in $H^{0,\delta}$ with $d/2 < \delta < \beta$.

$\because$) Set $v = \mathcal{F}\mathcal{M}_+ U(0,t)u$. Following H-N '98, from (NLS),

$$i\partial_t v(t) = |\zeta_2(t)|^{-\frac{1}{1-\lambda}} |v(t)|^{p_c} v(t) + o(t^{-1}).$$

1st term of r.h.s is $O(t^{-1})$, that is, non-integrable w.r.t. $t \in [r_0, \infty)$. Hence we introduce a phase modification $v_p = \mathcal{F}^{-1}\mathcal{P}(t)v$, where

$$\mathcal{P}(t) \coloneqq \exp\left(i\eta \int_{r_0}^{t} \left(|\tau|^{-\chi} + |v(\tau)|^2 \right)^{\frac{p_c}{2}} |\zeta_2(\tau)|^{-\frac{1}{1-\lambda}} \, d\tau \right), \ \chi > 0.$$

By the modification, we obtain

$$i\partial_t(\mathcal{F}v_p) = \underbrace{|\zeta_2(t)|^{-\frac{1}{1-\lambda}}}_{\sim |t|^{-1}} \mathcal{P}(t) \left(|v|^{p_c} - \left(|t|^{-\chi} + |v|^2 \right)^{\frac{p_c}{2}} \right) v + o(t^{-1}).$$

cf. Ginibre-Ozawa '93

Set $\chi > 0$, $n \in \mathbb{N} \cup \{0\}$, $t \geqslant r_0$. We denote

$$\mathscr{A}_n(t;z) = \left(|z|^{p_c - n} - \left(|t|^{-\chi} + |z|^2 \right)^{\frac{p_c - n}{2}} \right) z^n \quad (z \in \mathbb{C}).$$

Lemma 2

- If $p_c \in (0,2)$, then $|\mathscr{A}_n(t;z)| \lesssim |t|^{-\frac{p_c}{2}\chi}$.
- If $p_c \in (0,1)$, then $|\mathscr{A}_0(t;z)z| \lesssim |t|^{-p_c\chi}|z|^{1-p_c}$.

$\because$) The concavity of $|z|^{p_c}$ and the induction argument.

This implies that $\exists! u_+ \in H^{0,\delta}$ s.t. $v_p \to u_+$ in $H^{0,\delta}$ as $t \to \infty$.

Thank you for your attention.

Lemma 3

Let $d = 2, 3$. Assume $d/2 < \delta < \beta < \min(1 + p_c, d)$. Let u be the solution given by Proposition 2. Then

$$\|\mathcal{P}(\tau)\mathscr{A}(\tau)\widehat{v}(\tau)\|_{H^\delta} \lesssim \varepsilon\tau^{-\frac{\chi}{2}\kappa+2a(\varepsilon,\lambda)},$$

where $\mathscr{A}(t) = \mathscr{A}_0(t; \hat{v}(t))$ and $\kappa = 1 + p_c - \delta$ if $p_c \geqslant 1$,

$$\kappa = \min\left(3p_c - \delta, (1 - p_c)(1 + p_c - \delta) + p_c^2\right) \quad \text{if } p_c < 1.$$

$$\begin{aligned}\|\widehat{v_p}(t) - \widehat{v_p}(s)\|_{H^\delta} &\lesssim \int_s^t \|\mathcal{P}(\tau)\mathscr{A}(\tau)\widehat{v}(\tau)\|_{H^\delta}|\zeta_2(\tau)|^{-\frac{1}{1-\lambda}}\, d\tau + o(1) \\ &\lesssim \varepsilon t^{-c(\varepsilon,\lambda)} \quad (r_0 < s < t).\end{aligned}$$

Then $\exists! u_+ \in H^{0,\delta}$ s.t. $\|v_p(t) - u_+\|_{H^{0,\delta}} \lesssim \varepsilon t^{-c(\varepsilon,\lambda)} \quad (t > r_0)$.

Derivation of the integral equation

Key Tool: $U(t,0)=\mathcal{M}_1(t)\mathcal{D}_1(t)\mathcal{F}\mathcal{M}_+\mathcal{M}_2(t).$

Let us consider $\mathcal{E}(t)=-i\partial_t\left(U(0,t)u_p^-\right)+U(0,t)F(u_p^-)$. Set

$$u_p^-(t)=\mathcal{M}_1(t)\mathcal{D}_1(t)\widehat{w}(t),\quad \widehat{w}(t)=\exp\left(\frac{i\,|\widehat{u_-}|^{p_c}}{c_+}\log|t|\right)\widehat{u_-}.$$

Note that $i\partial_t\widehat{w}=(c_+|s|)^{-1}F(\widehat{w})$. By the factorization, we have

$$\begin{aligned}U(0,t)F(u_p^-)&=\mathcal{M}_2^{-1}(t)\mathcal{M}_+^{-1}\mathcal{F}^{-1}\left(|\zeta_2(t)|^{-\frac{1}{1-\lambda}}-(c_+|t|)^{-1}\right)F(\widehat{w})\\&\quad+\mathcal{M}_+^{-1}\left(\mathcal{M}_2^{-1}(t)-1\right)\mathcal{F}^{-1}(c_+|t|)^{-1}F(\widehat{w})\\&\quad+\mathcal{M}_+^{-1}\mathcal{F}^{-1}i\partial_t\widehat{w},\end{aligned}$$

$$i\partial_t\left(U(0,t)u_p^-\right)=i\partial_t\mathcal{M}_+^{-1}\left(\mathcal{M}_2^{-1}(t)-1\right)\mathcal{F}^{-1}\widehat{w}+\mathcal{M}_+^{-1}\mathcal{F}^{-1}i\partial_t\widehat{w}.$$

$$\mathcal{E}(t) = \mathcal{M}_+^{-1}\mathcal{M}_2^{-1}(t)\mathcal{F}^{-1}\left(|\zeta_2(t)|^{-\frac{1}{1-\lambda}} - (c_+|t|)^{-1}\right) F\left(\widehat{w}(t)\right)$$
$$- \mathcal{M}_+^{-1}\left(1 - \mathcal{M}_2^{-1}(t)\right)\mathcal{F}^{-1}(c_+|t|)^{-1}F\left(\widehat{w}(t)\right)$$
$$+ i\partial_t\mathcal{M}_+^{-1}\left(1 - \mathcal{M}_2^{-1}(t)\right)\mathcal{F}^{-1}\widehat{w}(t).$$

Then

$$\widetilde{\mathcal{E}}(t) = -i\int_{-\infty}^{t} U(t,0)\mathcal{M}_2(s)^{-1}\mathcal{M}_+^{-1}\mathcal{F}^{-1}$$
$$\times \underbrace{\left(\frac{c_+|s|}{|\zeta_2(s)|^{1/(1-\lambda)}} - 1\right)}_{\lesssim |s|^{\frac{-(1-2\lambda)}{1-\lambda}}} F(\widehat{w}(s))\,\frac{ds}{c_+|s|},$$
$$+ R(t)\widehat{w}(t) + i\int_{-\infty}^{t} U(t,s)R(s)F(\widehat{w}(s))\frac{ds}{c_+|s|},$$
$$R(t) := \mathcal{M}_1(t)\mathcal{D}_1(t)\underbrace{\left(\mathcal{F}\mathcal{M}_2(t)\mathcal{F}^{-1} - 1\right)}_{\lesssim \left||\nabla|^2 t^{-(1-2\lambda)}\right|^{(\alpha-\beta)/2}}.$$

Assumption 1 (e.g., Geluk-Marić-Tomić'93, Naito '84.)

$\exists\zeta_j(t)$: sol. to (1) satisfying the following: $\exists r_0 \geqslant 1$ s.t. $\forall |t| \geqslant r_0$,

(A1) $\exists\delta_0 > 0$ s.t. $|\zeta_2(t)| \geqslant \delta_0$,

(A2) $\exists a_j \in \mathbb{R}$ with $a_2 \neq 0$ and $\lambda < 1/2$ with $-1/3 < \lambda$ if $d = 3$ s.t.

$$\left|\frac{\zeta_2(t)}{|t|^{1-\lambda}} - a_2\right| + \left|\frac{\zeta_1(t)}{\zeta_2(t)} - \frac{a_1}{a_2}\right| \lesssim \begin{cases} |t|^{-(1-2\lambda)} \ (\lambda \geqslant 0), \\ |t|^{-1} \ (\lambda < 0), \end{cases} \tag{2}$$

$$\left|\frac{\zeta_2(t)}{|t|^{1-\lambda}} - a_2\right| \leqslant \frac{|a_2|}{2}. \tag{3}$$

NOTE: $\left|\mathcal{M}\left(\frac{\zeta_2(t)}{\zeta_1(t)}\right) - \mathcal{M}_+\right| = |\mathcal{M}_2(t) - 1| \lesssim |x|^{2\theta}\left|\frac{\zeta_1(t)}{\zeta_2(t)} - \frac{a_1}{a_2}\right|^{\theta}$.

(2) $\to |\zeta_2(t)| \lesssim |t|^{1-\lambda}$. (3) is only utilized to estimate $\mathcal{A}(t)$.

$\underline{p_c < 1}$: For any $s_0 \in (\beta - 1, p_c)$, G-N ineq. gives us

$$\begin{aligned} &\left\| |\nabla|^{\beta-1} \left(|\phi|^{p_c} - |\psi|^{p_c} \right) \right\|_{L^{\frac{2\beta}{\beta-1}}} \\ \lesssim &\, \|\phi - \psi\|_{L^\infty}^{p_c\left(1-\frac{\beta-1}{s_0}\right)} \left(\left\| |\nabla|^{s_0} |\phi|^{p_c} \right\|_{L^{\frac{2\beta}{s_0}}} + \left\| |\nabla|^{s_0} |\psi|^{p_c} \right\|_{L^{\frac{2\beta}{s_0}}} \right)^{\frac{\beta-1}{s_0}}. \end{aligned}$$

Lemma 4 (Vișan '07)

F: Hölder conti. of order $\rho \in (0,1)$. Suppose $\sigma \in (0, \rho)$, $p \in (1, \infty)$, $s \in (\sigma/\rho, 1)$, $1/p = 1/p_1 + 1/p_2$ and $(1 - \sigma/(\rho s))p_1 > 1$. Then

$$\left\| |\nabla|^\sigma F(u) \right\|_{L^p} \lesssim \left\| |u|^{\rho - \sigma/s} \right\|_{L^{p_1}} \left\| |\nabla|^s u \right\|_{L^{p_2 \sigma/s}}^{\sigma/s}.$$

$$\begin{aligned} \left\| |\nabla|^{s_0} |\varphi|^{p_c} \right\|_{\frac{2\beta}{s_0}} &\lesssim \|\varphi\|_{L^\infty}^{p_c - \frac{s_0}{s}} \left\| |\nabla|^s \varphi \right\|_{L^{\frac{2\beta}{s}}}^{\frac{s_0}{s}}, \quad s \in (s_0/p_c, 1), \\ \left\| |\nabla|^s \varphi \right\|_{L^{\frac{2\beta}{s}}} &\lesssim \|\varphi\|_{L^\infty}^{1-s} \|\nabla \varphi\|_{L^{2\beta}}^s. \end{aligned}$$

Remark 2

- In Proposition 1, we can replace the assumption $\|u_-\|_{H^{0,\alpha}} \ll 1$ by $\|\widehat{u_-}\|_\infty \ll 1$, but we need in the sequel the smallness in $H^{0,\alpha}$ to define $\mathcal{S}_+$.

Lemma 5

Let ε, A, $K > 0$, $u_0 \in L^2$ with $\|u_0\|_{L^2} + \|U(0, r_0)u_0\|_{H^{0,\beta}} \leqslant K$. Then $\exists T = T(K) > 0$ not depending on ε and A s.t. (NLS) has a unique sol. $u \in \mathscr{X}_T^{\varepsilon,A}$ with $u(r_0) = u_0$. Moreover

$$\|u\|_{\mathscr{X}_T^{\varepsilon,A}} \leqslant B_0 K, \quad B_0 > 0: \text{ not depending on } r_0,\ T,\ A,\ \varepsilon.$$

$$\mathscr{X}_T^{\varepsilon,A} := \{u \in C(I_T; L^2 \cap L^\infty(\mathbb{R}^d)) \mid \|u\|_{\mathscr{X}_T^{\varepsilon,A}} < \infty\}$$

equipped with $d(u, v) = \|u - v\|_{L^\infty(I_T;L^2)}$, where $I_T = [r_0, r_0 + T]$,

$$\|u\|_{\mathscr{X}_T^{\varepsilon,A}} := \sup_{t \in I_T} |t|^{-A\varepsilon^{\frac{1}{d(1-\lambda)}}} \|U(0,t)u\|_{H^{0,\beta}} + \sup_{t \in I_T} |t|^{\frac{d}{2}(1-\lambda)} \|u(t)\|_{L^\infty}.$$

$\because$) The contraction argument from the completeness of $(\mathscr{X}_T^{\varepsilon,A}, d)$.

By Lemma 5, we have $\|u\|_{\mathscr{X}_T^{\varepsilon,A}} < \varepsilon^{\frac{1}{2}}$, if $K = \varepsilon$ is small.

Lemma 6 (Bootstrap estimate)

Fix $\varepsilon \ll 1$. Let $T > 0$, $u_0 \in L^2$ with $\|u_0\|_{L^2} + \|U(0, r_0)u_0\|_{H^{0,\beta}} \leqslant \varepsilon$, and the solution $u \in \mathscr{X}_T^{\varepsilon,A}$ of (NLS) with $u(r_0) = u_0$ satisfies

$$\|u\|_{\mathscr{X}_T^{\varepsilon,A}} \leqslant \varepsilon^{\frac{1}{2}}.$$

Then $\exists b_1 > 0$ not depending on T and ε s.t. $\|u\|_{\mathscr{X}_T^{\varepsilon,A}} \leqslant b_1\varepsilon$.

$\because$) Thanks to the a priori bound, the desired assertion holds.

Proof of the global existence.

Once we obtain Lemma 6, the global existence of sol. can be proven by the standard continuation argument. $\square$

Proof of Lemma 1

We have the representation formula of $U(t,0)\phi$ as follows:

$$
\begin{aligned}
(U(t,0)\phi)(x) &= e^{-ia(t)x^2} \frac{1}{(4\pi i b(t))^{n/2}} \int_{\mathbb{R}^d} e^{i(x-y)^2/(4b(t))} e^{-ic(t)y^2} \phi(y)\, dy \\
&= e^{-i(a(t)-1/(4b(t)))x^2} \mathcal{D}(2b(t)) \mathcal{F}[\psi](x),
\end{aligned}
$$

where $\psi(y) = e^{i(1/(4b(t))-c(t))y^2}\phi(y)$, $\phi \in \mathscr{S}(\mathbb{R}^d)$, and

$$
a(t) = \frac{1-\zeta_2'(t)}{2\zeta_2(t)}, \quad b(t) = \frac{\zeta_2(t)}{2}, \quad c(t) = \frac{1-\zeta_1(t)}{2\zeta_2(t)}.
$$

Hence we have

$$
U(t,0) = \mathcal{M}\left(\frac{\zeta_2(t)}{\zeta_2'(t)}\right) \mathcal{D}\left(\zeta_2(t)\right) \mathcal{F} \mathcal{M}\left(\frac{\zeta_2(t)}{\zeta_1(t)}\right).
$$

□

Strichartz' estimate for $U(t,s)$

Lemma 7 (Kawamoto-Yoneyama'17, Kawamoto'20)

Let (q,r) and $(\tilde{q},\tilde{r})$ be admissible pairs and s' denotes the Hölder conjugate of s, i.e., $1/s+1/s'=1$. Then

$$\|U(t,0)\phi\|_{L^q(\mathbb{R}\,;\,L^r(\mathbb{R}^d),\lambda)} \lesssim \|\phi\|_{L^2},$$
$$\left\|\int_{\mathbb{R}} U(t,s)F(s)ds\right\|_{L^q(\mathbb{R}\,;\,L^r(\mathbb{R}^d),\lambda)} \lesssim \|F\|_{L^{\tilde{q}'}(\mathbb{R}\,;\,L^{\tilde{r}'}(\mathbb{R}^d),-\lambda)},$$

where (q,r), $(\tilde{q},\tilde{r})$: admissible pair $\overset{\text{def}}{\Leftrightarrow}$ $\frac{2}{q}=\frac{d}{2}-\frac{d}{r}$, $q\in(2,\infty]$,

$$\|F\|_{L^q(\mathbb{R}\,;\,L^r(\mathbb{R}^d),\lambda)} = \left(\int_{\mathbb{R}} \langle s\rangle^{-\lambda} \|F(s,\cdot)\|_{L^r}^q\, ds\right)^{1/q},$$
$$\|F\|_{L^\infty(\mathbb{R}\,;\,L^2(\mathbb{R}^d),\lambda)} = \sup_{s\in\mathbb{R}} \langle s\rangle^{-\lambda} \|F(s,\cdot)\|_{L^2}.$$